GQL INFERENCES IN LINEAR MIXED MODELS WITH DYNAMIC MEAN STRUCTURE

BINGRUI SUN

GQL Inferences in Linear Mixed Models with Dynamic Mean Structure

by

©Bingrui Sun

A thesis submitted to the
School of Graduate Studies
in partial fulfillment of the
requirements for Degree of
Master of Science in Statistics

Department of Mathematics and Statistics
Memorial University of Newfoundland

St. John's Newfoundland, Canada September 2009

Abstract

In some panel data studies for continuous data, the expectation of the response variable of an individual (or individual firm) at a given point of time may depend on the covariate history up to the present time. Also, the response at a given point of time may be influenced by an individual random effect. This type of data are usually analyzed by fitting a linear mixed model with dynamic mean structure. When the distribution of the random effects and error components of the model are not known, the likelihood inferences can not be used any longer. As a possible remedy, there exists some alternative estimation methods such as bias corrected least squares dummy variable (BCLSDV) and instrumental variables based generalized method of moments (IVGMM), which however may produce inefficient estimates. In this thesis, we develop a new GMM as well as a generalized quasi-likelihood (GQL) estimating approach and demonstrate that they perform well in estimating all parameters of the model, the GQL being in general more efficient than the GMM approach.

Acknowledgements

I would like to take this opportunity to acknowledge my supervisor Professor B.C. Sutradhar for suggesting this research problem, and for his constant encouragement, valuable suggestions and unbelievable patience throughout my program.

I would also like to recognize the School of Graduate Studies and the Department of Mathematics and Statistics for funding my graduate program. Also, I wish to thank all faculty and staff in the department for their kindness and great help.

I like to thank Drs. Gary Sneddon and Alwell Oyet for their comments and suggestions.

Many thanks go to my family for their unwavering support and encouragement.

It is also my great pleasure to thank my friends and fellow students for their suggestions and typesetting advice.

Contents

List of Tables

Chapter 1

Introduction

1.1 Background of the Problem

Panel data analysis is an important research problem in economics and biomedical fields, among others. In this set up, a small number of repeated responses are collected from a large number of independent individuals/firms/states. Let y_{it} be the tth ($t = 1, \ldots, T$) response of the ith ($i = 1, \ldots, K$) individual. Here T is usually small such as $T = 5$, and K is large, tending to infinity such as $K = 100$ or 200. Furthermore, a covariate vector is also collected from the ith individual at time point t. Let $x_{it} = (x_{it1}, \ldots, x_{itu}, \ldots, x_{itp})'$ be the $p-$ dimensional covariate vector corresponding to y_{it}. Note that $y_{i1}, \ldots, y_{it}, \ldots, y_{iT}$ along with $x_{i1}, \ldots, x_{it}, \ldots, x_{iT}$ constitute the panel data. It is expected that the repeated responses $y_{i1}, \ldots, y_{it}, \ldots, y_{iT}$ will be dynamically related and cause some autocorrelations among them. In this type of panel data set up, it is of primary interest to examine the effect of the covariates on the repeated responses after taking the dynamic dependence of the responses into account. For example, in an economic study, one may deal with repeated unemployment rates (y_{it}) over a period of 10 years, say, as a function of the associated economic growth rates (x_{it}) from the past. Here, to examine the effects of the growth rates on the unemployment rates, one needs to account for the dynamic dependence among the unemployment rates over the years. Similarly, in a biomedical field, one may study the effects of certain covariates (x_{it}) such as gender and education levels on the number of visits to the physician by an individual over a period of several years. Here, it is important to accommodate the longitudinal correlations of the repeated visits in finding the effects of the covariates on the responses.

Note that the panel or longitudinal data discussed above may be modeled as

$$y_{it} \;=\; x'_{it}\beta + \gamma y_{i,t-1} + \epsilon_{it}, \tag{1.1}$$

where $\beta = (\beta_1, \ldots, \beta_p)'$ is the effect of x_{it} on y_{it}, γ is the dynamic dependence parameter relating $y_{i,t-1}$ to y_{it}, and ϵ_{it} is an independently and identically distributed (iid) random error variable with mean 0 and variance σ_ϵ^2. It is standard to use the notation

$$\epsilon_{it} \overset{iid}{\sim} (0, \sigma_\epsilon^2). \tag{1.2}$$

Note that it is more practical to assume that the distribution of ϵ_{it} is not known, even though some authors have used the normality assumption in some economic models in the past. Further suppose that the response y_{it} is influenced by an unobservable random effect η_i^* , which is shared by all responses of the ith individual recorded over T periods of time. Let

$$\eta_i^* \overset{iid}{\sim} (0, \sigma_\eta^2). \tag{1.3}$$

Also, let ϵ_{it} and η_i^* be independent. One may then extend the dynamic fixed model (1.1) to the dynamic mixed model, given by

$$\begin{aligned} y_{i1} &= x'_{i1}\beta + \sigma_\eta \eta_i + \epsilon_{i1} \\ y_{it} &= x'_{it}\beta + \gamma y_{i,t-1} + \sigma_\eta \eta_i + \epsilon_{it}, \text{ for } t = 2, \ldots, T, \end{aligned} \tag{1.4}$$

where $\eta_i = \eta_i^*/\sigma_\eta$, and $\epsilon_{it} \overset{iid}{\sim} (0, \sigma_\epsilon^2)$.

It is of primary interest to fit the model (1.4) to a panel data set. This requires the estimation of the parameters of the model, namely, β, γ, σ_η^2, and σ_ϵ^2. The purpose of the thesis is to seek for a better, i.e, more efficient estimation method as compared to the existing estimation methods. Note that the consistent and efficient estimation of all parameters of the model (1.4), especially for σ_η^2 and σ_ϵ^2 may not be easy. Consequently, some econometricians concentrated on the estimation of the so-called main parameters β and γ, and used the so-called least squares dummy variable (LSDV) (or covariance variable (CV)) estimation method. For example, we refer to Bun and Carree(2005,equation (3)and(4)) and Hsiao (2003, section 4.2, p. 71). For convenience, the LSDV method is explained in brief in Chapter 2. Some other authors have used the so-called instrumental variables (IV) approach for the estimation of β and γ. For example, we refer to Anderson and Hsiao (1981), Amemiya and Macurdy (1986), Arellano and Bond (1991), and Arellano and Bover (1995). We also explain this IV approach in Chapter 2. There also exists an IV based generalized method of moment (GMM) approach which we explain in Chapter 2 as well. For details of the GMM approach, see for example, Hansen (1982), Amemiya and Macurdy (1986), and Arellano and Bond (1991).

Note that the LSDV approach does not yield unbiased and hence consistent estimates for β and γ. As a remedy, Bun and Carree (2005) have examined a bias-corrected LSDV (BCLSDV) approach for the consistent estimates of β and γ. Thus, all these existing approaches, namely, BCLSDV, IV and IV based GMM are known to produce consistent estimates for β and γ. It is however known that these moments based approaches may be inefficient. Moreover, these approaches avoid the estimation of the variance components, especially the estimation of σ_η^2, which may be of importance in its own right as it explains the variation of the data due to an individual's latent random effect. To overcome the inefficiency of the existing estimation methods, in the present thesis, we use a slightly different GMM approach than the existing IV based GMM approach, not only for β and γ, but also for the variance component σ_η^2 and σ_ϵ^2. Following the suggestion of Rao, Sutradhar, and Pandit (2008), we also examine the efficiency performance of a generalized quasi-likelihood approach (GQL) for the estimation of all parameters.

1.2 Objective of the Thesis

The major objectives of the thesis are as follows: (a) We develop a new GMM and GQL estimation approach for all parameters of the dynamic mixed model (1.4); (b) We show that the GQL approach asymptotically always produces more efficient estimates than the GMM approach; (c) We also examine the small sample (say, K=100) performance of the GMM and GQL estimates for a wide range of values of the parameters, namely for β, γ, σ_η^2, and σ_ϵ^2. It is shown that while GMM produces unbiased estimates, it however gives inefficient estimates as compared to the GQL estimates.

Chapter 2

Estimation Methods

2.1 Existing Estimation Methods

2.1.1 LSDV and Bias Corrected LSDV Estimators

The LSDV estimators of the regression parameter β and the dynamic dependence parameter γ in model (1.4) can be derived by applying the least squares technique to a transformed model, where the transformation is achieved from (1.4) by using the deviations of variables from their individual-specific means. Let $\bar{y}_i = \frac{\sum_{t=1}^{T} y_{it}}{T}$, $\bar{y}_{i,-1} = \frac{\sum_{t=1}^{T} y_{i,t-1}}{T}$ for known y_{i0}, $\bar{x}_i = \frac{\sum_{t=1}^{T} x_{it}}{T}$, and $\bar{\epsilon}_i = \frac{\sum_{t=1}^{T} \epsilon_{it}}{T}$, and for $t = 1, \ldots, T$, $\tilde{y}_{it} = y_{it} - \bar{y}_i$, $\tilde{y}_{i,t-1} = y_{i,t-1} - \bar{y}_{i,-1}$, $\tilde{x}_{it} = x_{it} - \bar{x}_i$, $\tilde{\epsilon}_{it} = \epsilon_{it} - \bar{\epsilon}_i$. We may then write the transformed model from (1.4) as:

$$\tilde{y}_{it} = \tilde{x}_{it'}\beta + \gamma \tilde{y}_{i,t-1} + \tilde{\epsilon}_{it}, i = 1, \ldots, K. \tag{2.1}$$

Note that as opposed to the model (1.4), this model (2.1) is free from individual random effects η_i. Next, define $\tilde{y}_i = (y_{i1}, \ldots, y_{iT})'$,
$\tilde{x}_i = (\tilde{x}_{i1}, \ldots, \tilde{x}_{it}, \ldots, \tilde{x}_{iT})'_{:T\times p}$ and $X_i^* = (\tilde{x}_i, \tilde{y}_{i,-1})_{:T\times(p+1)}$, with
$\tilde{y}_{i,-1} = (y_{i0}, \ldots, y_{i,T-1})'$. Now, by applying the well known least squares method to

the transformed model (2.1), one obtains the LSDV estimator for $(\beta, \gamma)'$ as

$$\begin{pmatrix} \hat{\beta}_{LSDV} \\ \hat{\gamma}_{LSDV} \end{pmatrix} = \left(\sum_{i=1}^{K} X_i^{*\prime} X_i^{*}\right)^{-1} \sum_{i=1}^{K} X_i^{*\prime} \tilde{y}_i \tag{2.2}$$

Note that in the economic literature, many authors, for example Bun and Carree (2005) and Hsiao (2003) have used this LSDV technique for the estimation of the parameters. But, it is well known that these estimators are not unbiased. As a remedy, Bun and Carree (2005) have done some corrections to reduce the bias. However, as the covariances or the correlations of the data were not accommodated in such LSDV technique, it is most likely to give a different estimate. In the thesis, especially in Section 2.2, we provide some new techniques of estimation, they are developed by accommodating the correlation of the data.

2.1.2 Instrumental Variables Method

For the analysis of the correlated data following model (1.4), some econometricians (Amemiya and MaCurdy (1986)) have used an instrumental variables based least squares technique. We explain this technique below in brief.

By taking the first difference based on (1.4), one writes

$$y_{it} - y_{i,t-1} = (x'_{it} - x'_{i,t-1})\beta + \gamma(y_{i,t-1} - y_{i,t-2}) + \epsilon_{it} - \epsilon_{i,t-1}, \text{ for } t = 3, \ldots, T. \tag{2.3}$$

Because $(y_{i,t-2} - y_{i,t-3})$, $(y_{i,t-3} - y_{i,t-4})$, $\ldots$ are correlated with $(y_{i,t-1} - y_{i,t-2})$ but are uncorrelated with $(\epsilon_{it} - \epsilon_{i,t-1})$, it is expected that past difference will provide some more information about the paremeters of the model, especially for γ. This is one of the main reasons why $(y_{i,t-2} - y_{i,t-3})$, $(y_{i,t-3} - y_{i,t-4})$, $\ldots$ are considered as instruments for $(y_{i,t-1} - y_{i,t-2})$. In this approach, these instrumental variables (IV) are used to develop a least squares estimation technique. Thus the IV estimates of γ and β are given by

$$\begin{pmatrix} \hat{\gamma}_{iv} \\ \hat{\beta}_{iv} \end{pmatrix} = \left[\sum_{i=1}^{K}\sum_{t=3}^{T} \begin{pmatrix} (y_{i,t-1} - y_{i,t-2})(y_{i,t-2} - y_{i,t-3}) & (y_{i,t-2} - y_{i,t-3})(x_{it} - x_{i,t-1})' \\ (x_{it} - x_{i,t-1})(y_{i,t-2} - y_{i,t-3}) & (x_{it} - x_{i,t-1})(x_{it} - x_{i,t-1})' \end{pmatrix}\right]^{-1}$$

$$\times \left[\sum_{i=1}^{K} \sum_{t=3}^{T} \begin{pmatrix} (y_{i,t-2} - y_{i,t-3}) \\ (x_{it} - x_{i,t-1}) \end{pmatrix} (y_{it} - y_{i,t-1}) \right]. \tag{2.4}$$

These estimates are consistent.

As far as the estimation of the remaining parameters σ_η^2 and σ_ϵ^2 is concerned, they are estimated by the method of moments and these IV estimates are computed as follows:

$$\hat{\sigma}^2_{\epsilon,iv} = \frac{\sum_{i=1}^K \sum_{t=2}^T [(y_{it} - y_{i,t-1}) - \hat{\gamma}(y_{i,t-1} - y_{i,t-2}) - (x_{it} - x_{i,t-1})'\hat{\beta}]^2}{2K(T-1)},$$

and

$$\hat{\sigma}^2_{\eta,iv} = \frac{\sum_{i=1}^K (\bar{y}_i - \hat{\gamma}\bar{y}_{i,-1} - \bar{x}_i'\hat{\beta})^2}{K} - \frac{\hat{\sigma}_\epsilon^2}{T}. \tag{2.5}$$

The instrumental variables estimators of γ, β and σ_ϵ^2 are consistent when K or T or both tend to infinity. The estimator of σ_η^2 is consistent only when K goes to infinity, it is inconsistent if K is fixed and T tends to infinity. Note that when true σ_{eta}^2 is small, $\hat{\sigma}_\eta^2$ obtained by (2.5) may be negative in some occasions. In such a case, one uses $\hat{\sigma}_\eta^2 = 0$ to avoid the negative estimation. We remark, however, that the use of $\hat{\sigma}_\eta^2 = 0$ is not a proper solution, as this requires further investigation by other non-negative variance estimation methods.

It is noted by Amemiya and MaCurdy (1986), Arellano and Bond (1991), etc. that all $y_{i,t-2-j}$, $j = 0, 1, \ldots,$ are eligible instruments for $(y_{i,t-1} - y_{i,t-2})$ too, because they satisfy the conditions $E[y_{i,t-2-j}(y_{i,t-1} - y_{i,t-2})] \neq 0$ and $E[y_{i,t-2-j}(\epsilon_{it} - \epsilon_{i,t-1})] = 0$. Consequently, similar formulas as in (2.4) and (2.5) can be developed by using $y_{i,t-2-j}$ in place of $(y_{i,t-2} - y_{i,t-3})$, and so on, but we do not give it here for simplicity.

2.1.3 Instrumental Variables Based Generalized Method of Moments Estimators

By using the property that the instrumental variables are correlated with $(y_{i,t-1} - y_{i,t-2})$ and are uncorrelated with $(\epsilon_{it} - \epsilon_{i,t-1})$ in (2.3), one can write moment conditions (moment estimating equations) given by

$$EW_i \triangle \epsilon_i = 0 \tag{2.6}$$

where

$$W_i = \begin{bmatrix} q_{i3} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & q_{i4} & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \cdots & \cdots & \cdots & q_{it} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & q_{iT} \end{bmatrix}_{:[(T-2)(T-1)(p+\frac{1}{2})]\times(T-2)} \tag{2.7}$$

with $q_{it} = (y_{i1} - y_{i0}, y_{i2} - y_{i1}, \ldots, y_{i,t-2} - y_{i,t-3}, \triangle x_i')'$ for known y_{i0}, where $\triangle x_i' = [(x_{i2} - x_{i1})', \ldots, (x_{iT} - x_{i,T-1})']$. In (2.6), $\triangle\epsilon_i = [(\epsilon_{i3} - \epsilon_{i2}), \ldots, (\epsilon_{iT} - \epsilon_{i,T-1})]'$. Thus, $(T-2)(T-1)(p+\frac{1}{2})$ number of moment equations can be constructed to solve for the $(p+1)-$dimensional θ vector, which naturally would not yield any solutions. As a remedy, a generalized distance function such as

$$(\frac{1}{K}\sum_{i=1}^{K}\triangle\epsilon_i'W_i')\Phi^{-1}(\frac{1}{K}\sum_{i=1}^{K}W_i\triangle\epsilon_i) \tag{2.8}$$

is minimized with respect to θ, yielding $(p+1)$ equations to solve. Here $\Phi = E[1/K^2\sum_{i=1}^{N}W_i\triangle\epsilon_i\triangle\epsilon_i'W_i']$. The solution of these $(p+1)$ equations are known as the generalized method of moments (GMM) estimates.

Note that in the next section, we provide a new GMM where instrumental variables are firstly used in a different way to construct $(p+1)$ ordinary moment equations. The covariance matrix of the moment functions involved in the ordinary moment equations is then used as a weight matrix to construct a generalized distance function, which is different than (2.8).

2.2 Proposed GMM and GQL Methods

2.2.1 New GMM Method

Note that as opposed to the IV based GMM approach discussed in section 2.1.3, in this section we first form different moment conditions than those in (2.6) and then minimize the proposed distance functions almost in the same way as in (2.8). To be specific, our moment conditions will be involved in the ordinary method of moments estimating equations (MMEE) for the respective parameters. In order to write the MMEEs, we use the following lemma to describe the basic moment properties, specifically the first and second order moments of the model (1.4).

Lemma 1. Under the dynamic mixed model (1.4), the mean and variance of y_{it} $(t = 1, \dots, T)$ are given by

$$\begin{aligned} E(Y_{it}) = \mu_{it} &= \sum_{j=0}^{t-1} \gamma^j x'_{i,t-j}\beta, \text{ and} \qquad (2.9) \\ \text{var}(Y_{it}) = \sigma_{itt} &= \sigma_\eta^2 \left\{ \sum_{j=0}^{t-1} \gamma^j \right\}^2 + \sigma_\epsilon^2 \sum_{j=0}^{t-1} \gamma^{2j}, \qquad (2.10) \end{aligned}$$

respectively, and the auto-covariance of lag $t-u$ for $u < t$, is given by

$$\text{cov}(Y_{iu}, Y_{it}) = \sigma_\eta^2 \sum_{j=0}^{t-1} \gamma^j \sum_{k=0}^{u-1} \gamma^k + \sigma_\epsilon^2 \sum_{j=0}^{u-1} \gamma^{t-u+2j}. \qquad (2.11)$$

Proof: We write

$$z_{it} = x'_{it}\beta + \sigma_\eta \eta_i + \epsilon_{it}, \; t = 1, \dots, T \qquad (2.12)$$

Therefore, we have

$$\begin{aligned} y_{i1} &= z_{i1} \\ y_{it} &= \gamma y_{i,t-1} + z_{it}, \; t = 2, \dots, T \qquad (2.13) \\ &= \gamma(\gamma y_{i,t-2} + z_{i,t-1}) + z_{it} \\ &= \gamma^2 y_{i,t-2} + \gamma z_{i,t-1} + z_{it} \end{aligned}$$

$$
\begin{aligned}
&= \cdots \\
&= \gamma^{t-u} y_{iu} + \gamma^{t-u-1} z_{i,u+1} + \cdots + \gamma z_{i,t-1} + z_{it} && (2.14) \\
&= \cdots \\
&= \gamma^{t-1} y_{i1} + \gamma^{t-2} z_{i,2} + \cdots + \gamma z_{i,t-1} + z_{it} \\
&= \gamma^{t-1} z_{i1} + \gamma^{t-2} z_{i,2} + \cdots + \gamma z_{i,t-1} + z_{it} \\
&= \sum_{j=0}^{t-1} \gamma^j z_{i,t-j} && (2.15)
\end{aligned}
$$

Since $\eta_i \overset{iid}{\sim} (0,1)$, $\epsilon_{it} \overset{iid}{\sim} (0, \sigma_\epsilon^2)$, and η_i is independent of ϵ_{it}, it then follows that

$$E(Y_{it}) = \sum_{j=0}^{t-1} \gamma^j E(z_{i,t-j}) = \sum_{j=0}^{t-1} \gamma^j x'_{i,t-j}\beta$$

which is the same as equation (2.9), and

$$
\begin{aligned}
\mathrm{var}(y_{it}) &= \mathrm{var}(\sum_{j=0}^{t-1} \gamma^j z_{i,t-j}) \\
&= \mathrm{var}\left(\sum_{j=0}^{t-1} \gamma^j (x'_{i,t-j}\beta + \sigma_\eta \eta_i + \epsilon_{i,t-j})\right) \\
&= \mathrm{var}\left((\sigma_\eta \sum_{j=0}^{t-1} \gamma^j)\eta_i\right) + \mathrm{var}(\sum_{j=0}^{t-1} \gamma^j \epsilon_{i,t-j}) \\
&= \sigma_\eta^2 (\sum_{j=0}^{t-1} \gamma^j)^2 + \sigma_\epsilon^2 \sum_{j=0}^{t-1} \gamma^{2j} && (2.16)
\end{aligned}
$$

which is the same as equation (2.10). Next for $u < t$, it follows from (2.14) that

$$y_{it} = \gamma^{t-u} y_{i,u} + \sum_{j=0}^{t-u-1} \gamma^j z_{i,t-j} \tag{2.17}$$

so the covariance between y_{iu} and y_{it} is given by

$$
\begin{aligned}
\sigma_{iut} &= \mathrm{cov}(y_{iu}, y_{it}) \\
&= \mathrm{cov}(y_{iu}, \gamma^{t-u} y_{iu} + \sum_{j=0}^{t-u-1} \gamma^j z_{i,t-j}) \\
&= \gamma^{t-u} \mathrm{var}(y_{iu}) + \sum_{j=0}^{t-u-1} \gamma^j \mathrm{cov}(y_{iu}, z_{i,t-j}). && (2.18)
\end{aligned}
$$

Note that as

$$\begin{aligned}
\mathrm{cov}(y_{iu}, z_{i,t-j}) &= \mathrm{cov}(\sum_{j=0}^{u-1} \gamma^j z_{i,u-j}, z_{i,t-j}) \\
&= \mathrm{cov}\left(\sum_{j=0}^{u-1} \gamma^j (x'_{i,u-j}\beta + \sigma_\eta \eta_i + \epsilon_{i,u-j}), x'_{i,t-j}\beta + \sigma_\eta \eta_i + \epsilon_{i,t-j}\right) \\
&= \sum_{j=0}^{u-1} \gamma^j \mathrm{cov}(x'_{i,u-j}\beta + \sigma_\eta \eta_i + \epsilon_{i,u-j}, x'_{i,t-j}\beta + \sigma_\eta \eta_i + \epsilon_{i,t-j}) \\
&= \sum_{j=0}^{u-1} \gamma^j \mathrm{cov}(\sigma_\eta \eta_i, \sigma_\eta \eta_i) \\
&= \sum_{j=0}^{u-1} \gamma^j \sigma_\eta^2, \qquad (2.19)
\end{aligned}$$

by using (2.16) and (2.19) in (2.18), we obtain

$$\begin{aligned}
\sigma_{iut} &= \gamma^{t-u}\left(\sigma_\eta^2 (\sum_{j=0}^{u-1} \gamma^j)^2 + \sigma_\epsilon^2 \sum_{j=0}^{u-1} \gamma^{2j}\right) + \sigma_\eta^2 \sum_{j=0}^{t-u-1} \gamma^j \sum_{j=0}^{u-1} \gamma^j \\
&= \sigma_\epsilon^2 \sum_{j=0}^{u-1} \gamma^{t-u+2j} + \sigma_\eta^2 \sum_{j=0}^{u-1} \gamma^j \left(\sum_{j=0}^{u-1} \gamma^{t-u+j} + \sum_{j=0}^{t-u-1} \gamma^j\right) \\
&= \sigma_\eta^2 \sum_{j=0}^{t-1} \gamma^j \sum_{k=0}^{u-1} \gamma^k + \sigma_\epsilon^2 \sum_{j=0}^{u-1} \gamma^{t-u+2j}, \qquad (2.20)
\end{aligned}$$

which is the same as the equation (2.11).

Note that β is known to be a location parameter, whereas γ is a dependence or correlation parameter, and σ_η^2 and σ_ϵ^2 are scale parameters. Consequently, in the thesis, we estimate these three types of parameters using three similar but different estimating equations. The estimation process of β will be explained in detail, but the estimation of γ and $\alpha = (\sigma_\eta^2, \sigma_\epsilon^2)'$ will be given in brief, for there are a lot of similarities in the theoretic deviations of their estimating equations. In order to develop the estimating equations for each type of parameter, we first construct a moment function so that its expectation involves the parameter of interest. Secondly, we construct a generalized distance function and use a minimization technique to obtain the respective estimates. We also discuss the asymptotic and small sample properties of the estimates.

GMM Estimation of β

To develop the GMM framework (Hansen (1982)) for the estimation of β involved in model (1.4), we first note by (2.9) that β is contained in the first moment of y_{it} only, for all $i = 1, \ldots, K$ and $t = 1, \ldots, T$. So we first write a sample based moment function given by

$$W_1 = \sum_{i=1}^{K}\sum_{t=1}^{T}\left(\sum_{j=0}^{t-1}\gamma^j x_{i,t-j}\right) y_{it} \tag{2.21}$$

as a reflection of this first order moment keeping in mind that β is a vector parameter. It then follows that

$$\begin{aligned} E(W_1) &= \sum_{i=1}^{K}\sum_{t=1}^{T}\left(\sum_{j=0}^{t-1}\gamma^j x_{i,t-j}\right) \mu_{it} \\ &= \sum_{i=1}^{K}\sum_{t=1}^{T}\left(\sum_{j=1}^{t-1}\gamma^j x_{i,t-j}\right)\left(\sum_{j=0}^{t-1}\gamma^j x'_{i,t-j}\beta\right) \end{aligned} \tag{2.22}$$

by (2.9). Secondly, we write an ordinary distance function

$$\psi_1 = W_1 - E(W_1), \tag{2.23}$$

so that $E[\psi_1] = 0$.

Now to estimate β involved in $E(W_1)$, we minimize a generalized distance function given by

$$Q_1 = \psi_1' C_1 \psi_1 \tag{2.24}$$

instead of solving the ordinary distance function (2.23), where C_1 is a suitable $p \times p$ positive definite matrix, with $C_1 = [\mathrm{cov}(\psi_1)]^{-1}$ as the optimal choice under a class of moments based estimation. Note that minimizing the distance function in (2.24) is equivalent to solve the estimating equation

$$\frac{\partial \psi_1'}{\partial \beta} C_1 \psi_1 = 0, \tag{2.25}$$

which is referred to as the GMM estimating equation for β. In (2.25), $\frac{\partial \psi_1'}{\partial \beta}$ is the first order derivative of ψ_1' with respect to β. Now to solve for β, we use the Gauss-Newton iterative equation

$$\hat{\beta}_{GMM,(r+1)} = \hat{\beta}_{GMM,(r)} + \left[\frac{\partial \psi_1'}{\partial \beta} C_1 \frac{\partial \psi_1}{\partial \beta'}\right]^{-1}_{(r)} \left[\frac{\partial \psi_1'}{\partial \beta} C_1 \psi_1\right]_{(r)}, \tag{2.26}$$

where $(\ )_r$ denotes that the expression within the square bracket is evaluated at $\beta = \hat{\beta}_{GMM,(r)}$, the estimate obtained for the r-th iteration. Let the final solution obtained from (2.26) be denoted by $\hat{\beta}_{GMM}$. Under some mild regularity conditions it may be shown that as $K \to \infty$,

$$K^{\frac{1}{2}}(\hat{\beta}_{GMM} - \beta) \sim N\left[0, K\left\{\frac{\partial \psi_1'}{\partial \beta} C_1 \frac{\partial \psi_1}{\partial \beta}\right\}^{-1} \left(\frac{\partial \psi_1'}{\partial \beta} C_1 \frac{\partial \psi_1}{\partial \beta'}\right) \left\{\frac{\partial \psi_1'}{\partial \beta} C_1 \frac{\partial \psi_1}{\partial \beta'}\right\}^{-1}\right] \tag{2.27}$$

[Hsiao (2003, eqn. (3.4.2), p. 65)], where $C_1^{-1} = \text{cov}(\psi_1)$ is the true covariance matrix for ψ_1 based on the true data. At the end, it follows that the covariance matrix of $\hat{\beta}_{GMM}$ is

$$\text{cov}(\hat{\beta}_{GMM}) = \left\{\frac{\partial \psi_1'}{\partial \beta} C_1 \frac{\partial \psi_1}{\partial \beta'}\right\}^{-1}. \tag{2.28}$$

GMM Estimation of γ

For estimation of γ involved in model (1.4), we first note by (2.9), (2.10) and (2.11) that γ is contained in the first and second moments of y_{it}. So we firstly write a moment function given by

$$W_2 = \sum_{i=1}^{K} \sum_{t=1}^{T-1} (y_{it} - \mu_{it})(y_{i,t+1} - \mu_{i,t+1}) \tag{2.29}$$

as a reflection of the first and second order moments keeping in mind that γ is a scale parameter. By taking the expectation over (2.29), we obtain

$$\begin{aligned} E(W_2) &= \sum_{i=1}^{K} \sum_{t=1}^{T-1} \text{cov}(y_{it}, y_{i,t+1}) \\ &= \sum_{i=1}^{K} \sum_{t=1}^{T-1} \sigma_{it,t+1}, \end{aligned} \tag{2.30}$$

which contains γ and other scale parameters by (2.11). We then write an ordinary distance function

$$\psi_2 = W_2 - E(W_2), \tag{2.31}$$

so that $E[\psi_2] = 0$, and attempt to estimate γ by minimizing a generalized distance function given by

$$Q_2 = \psi_2^2 C_2 \tag{2.32}$$

with $C_2 = [\text{var}(\psi_2)]^{-1}$ as the optimal choice. Note however that unlike the computation of C_1 in (2.24), the computation of C_2 in (2.32) requires the formula for the fourth order moments of the responses. But, as the errors and the random effects in model (1.4) do not necessarily have normal distributions, one, therefore, can not compute C_2 without knowing the true distributions of the errors and the random effects.

To avoid the above distributional difficulty, we choose to minimize a 'working' generalized distance function, namely,

$$Q_2 = \psi_2^2 C_{2N}, \tag{2.33}$$

where C_{2N}^{-1} is a 'normality' based variance of ψ_2. Consequently, we solve the moment estimating equation

$$\frac{\partial \psi_2}{\partial \gamma} C_{2N} \psi_2 = 0, \tag{2.34}$$

for γ, which can be iteratively solved by

$$\hat{\gamma}_{GMM,(r+1)} = \hat{\gamma}_{GMM,(r)} + \left[\frac{\frac{\partial \psi_2}{\partial \gamma} C_{2N} \psi_2}{\frac{\partial \psi_2}{\partial \gamma} C_{2N} \frac{\partial \psi_2}{\partial \gamma}} \right]_{(r)}. \tag{2.35}$$

In the fashion similar to that of (2.27), as $K \to \infty$, we obtain the asymptotic distribution of $\hat{\gamma}_{GMM}$ as

$$K^{\frac{1}{2}}(\hat{\gamma}_{GMM} - \gamma) \sim N\left[0, K\left\{ \frac{\frac{\partial \psi_2}{\partial \gamma} C_{2N} C_2^{-1} C_{2N} \frac{\partial \psi_2}{\partial \gamma}}{(\frac{\partial \psi_2}{\partial \gamma} C_{2N} \frac{\partial \psi_2}{\partial \gamma})^2} \right\}\right], \tag{2.36}$$

where $C_2^{-1} = \text{var}(\psi_2)$ is the true variance for ψ_2 based on the true data. Note that (2.36) has a slightly different form than that of (2.27). This is because unlike for γ

estimation, in computing $\hat{\beta}$, one does not require the normality assumption for the errors and the random effects. Further note that if the true distributions of the errors and random effects were normal, then $C_2 = C_{2N}$. This leads to the variance of $\hat{\gamma}_{GMM}$ as

$$\text{var}(\hat{\gamma}_{GMM}) = \left\{ \frac{\partial \psi_2}{\partial \gamma} C_{2N} \frac{\partial \psi_2}{\partial \gamma} \right\}^{-1}, \tag{2.37}$$

which naturally is similar to (2.28).

GMM Estimation of $\alpha = (\sigma_\eta^2, \sigma_\epsilon^2)'$

To develop the GMM framework for the estimation of σ_η^2 and σ_ϵ^2, we first note by (2.10) and (2.11) that σ_η^2 and σ_ϵ^2 are contained in the variance and covariance of y_{it}. For the estimation of these two parameters, we use the following two moment functions given by

$$W_3 = \sum_{i=1}^{K} \sum_{t=1}^{T} (y_{it} - \mu_{it})^2 \tag{2.38}$$

and

$$W_4 = \sum_{i=1}^{K} \sum_{u=1}^{T-1} \sum_{t=u+1}^{T} (y_{iu} - \mu_{iu})(y_{it} - \mu_{it}) \tag{2.39}$$

for σ_η^2 and σ_ϵ^2, respectively. Since

$$\begin{aligned} E(W_3) &= \sum_{i=1}^{K} \sum_{t=1}^{T} \text{var}(y_{it}) \\ &= \sum_{i=1}^{K} \sum_{t=1}^{T} \sigma_{itt}, \end{aligned} \tag{2.40}$$

and

$$\begin{aligned} E(W_4) &= \sum_{i=1}^{K} \sum_{u=1}^{T-1} \sum_{t=u+1}^{T} \text{cov}(y_{iu}, y_{it}) \\ &= \sum_{i=1}^{K} \sum_{u=1}^{T-1} \sum_{t=u+1}^{T} \sigma_{iut}, \end{aligned} \tag{2.41}$$

where σ_{itt} and σ_{iut} are given by equations (2.10) and (2.11) respectively, we can now write two appropriate distance functions as $\psi_3 = W_3 - E(W_3)$ and $\psi_4 = W_4 - E(W_4)$

for σ_η^2 and σ_ϵ^2, respectively. Let $\psi_3^*=(\psi_3,\psi_4)'$. Since under the true model (1.4), the covariance of ψ_3^* requires the fourth order moments, similar to (2.34), we now solve the moment estimating equation for $\alpha = (\sigma_\eta^2, \sigma_\epsilon^2)'$ given by

$$\frac{\partial \psi_3^{*\prime}}{\partial \alpha} C_{3N} \psi_3^* = 0, \tag{2.42}$$

where C_{3N}^{-1} is the 'normality' based covariance of ψ_3^*. Similar to (2.27), as $K \to \infty$,

$$K^{\frac{1}{2}}(\hat{\alpha}_{GMM}-\alpha) \sim N\left[0, K\left\{\frac{\partial \psi_3^{*\prime}}{\partial \alpha} C_{3N} \frac{\partial \psi_3^*}{\partial \alpha}\right\}^{-1} \left(\frac{\partial \psi_3^{*\prime}}{\partial \alpha} C_{3N} C_3^{-1} C_{3N} \frac{\partial \psi_3^*}{\partial \alpha'}\right) \left\{\frac{\partial \psi_3^{*\prime}}{\partial \alpha} C_{3N} \frac{\partial \psi_3^*}{\partial \alpha'}\right\}^{-1}\right], \tag{2.43}$$

where $C_3^{-1} = \text{cov}(\psi_3^*)$ is the true variance for ψ_3^* based on the true data. Note that if the true distributions of the errors and random effects were normal, then $C_3 = C_{3N}$. This leads to the variance of $\hat{\alpha}_{GMM}$ as

$$\text{var}(\hat{\alpha}_{GMM}) = \left\{\frac{\partial \psi_3^{*\prime}}{\partial \alpha} C_{3N} \frac{\partial \psi_3^*}{\partial \alpha'}\right\}^{-1}. \tag{2.44}$$

2.2.2 Proposed Generalized Quasi-likelihood Estimation

In this section we develop a generalized quasi-likelihood (GQL) approach for the estimation of the parameters of the model (1.4). In the GQL approach, the regression effects β will be estimated following the GQL estimating equation suggested by Sutradhar (2003, Section 3), where the first and second order moments of the responses are exploited. For the GQL estimation of the dynamic dependence parameter (γ) and the variance components (σ_η^2 and σ_ϵ^2) of the model (1.4), we will follow the GQL estimating equations suggested by Sutradhar (2004, eqn. (3.4)) where the second, third and fourth order moments of the responses are utilized. Similar to what we did for the GMM approach, the estimation of parameters in this section is done in groups: first for β, then for γ, and lastly for $\alpha = (\sigma_\eta^2, \sigma_\epsilon^2)'$.

GQL Estimation of β

Recall that β is contained only in the means of responses. One may therefore construct a basic sufficient statistic based on the first order responses for the estimation of this parameter. To be specific, let $y_i = (y_{i1}, \ldots, y_{it}, \ldots, y_{iT})'$ be the $T \times 1$ vector of the first order responses for the ith individual. Also, let $\mu_i = E(Y_i) = (\mu_{i1}, \ldots, \mu_{it}, \ldots, \mu_{iT})'$ be the mean of y_i vector with $\mu_{it} = \sum_{j=0}^{t-1} \gamma^j x'_{i,t-j}\beta$ (2.9) for all $t = 1, \ldots, T$. Following Sutradhar (2003, Section 3), we may then write a generalized quasi-likelihood (GQL) estimating equation for β given by

$$\sum_{i=1}^{K} \frac{\partial \mu_i'}{\partial \beta} \Sigma_i^{-1}(y_i - \mu_i) = 0, \tag{2.45}$$

which is referred to as the GQL estimating equation for β, where $\Sigma_i = \text{var}(Y_i) = (\sigma_{iut})$ is the $T \times T$ true covariance matrix of y_i, with σ_{itt} and σ_{iut} given by (2.10) and (2.11) under Lemma 1. The GQL estimator of β can be solved by using the Gauss-Newton iterative equation

$$\hat{\beta}_{GQL,(r+1)} = \hat{\beta}_{GQL,(r)} + \left[\sum_{i=1}^{K} \frac{\partial \mu_i'}{\partial \beta} \Sigma_i^{-1} \frac{\partial \mu_i}{\partial \beta'}\right]_{(r)}^{-1} \left[\sum_{i=1}^{K} \frac{\partial \mu_i'}{\partial \beta} \Sigma_i^{-1}(y_i - \mu_i)\right]_{(r)}. \tag{2.46}$$

Since K individuals are independent, it follows by applying the standard central limit theory that asymptotically ($K \to \infty$), $\hat{\beta}_{GQL}$ has the multivariate normal distribution given by

$$\sqrt{K}(\hat{\beta}_{GQL} - \beta) \sim N\left(0, K\left[\sum_{i=1}^{K} \frac{\partial \mu_i'}{\partial \beta} \Sigma_i^{-1} \frac{\partial \mu_i}{\partial \beta'}\right]^{-1}\right), \tag{2.47}$$

[Hsiao, 2003, eqn. 3.4.2 p. 65]. Note that the GQL estimating equation in (2.45) is a generalization of the QL estimating equation for independent data, introduced by Wedderburn (1974) [see also McCullagh (1983)].

GQL Estimation of γ

Under the present model (1.4), γ is known as the dynamic dependence parameter. Since γ relates y_{it} to $y_{i,t-1}$ through the dynamic model (1.4), we write an elementary sufficient statistic vector consisting of the corrected lag-1 pair-wise products of the responses, given by

$$\begin{aligned} f_i &= [(y_{i1}-\mu_{i1})(y_{i2}-\mu_{i2}),\cdots,(y_{iu}-\mu_{iu})(y_{i,u+1}-\mu_{i,u+1}),\cdots, \\ &\quad (y_{i,T-1}-\mu_{i,T-1})(y_{iT}-\mu_{iT})]', \end{aligned} \tag{2.48}$$

assuming that $\mu_{it}^{'}$s are known. It then follows that

$$\lambda_i = E(f_i) = (\sigma_{i12},\cdots,\sigma_{it,t+1},\cdots,\sigma_{i,T-1,T})',$$

where $\sigma_{it,t+1} = \sigma_\eta^2 \sum_{j=0}^{t-1}\gamma^j \sum_{k=0}^{t}\gamma^k + \sigma_\epsilon^2 \sum_{j=0}^{t-1}\gamma^{1+2j}$ by (2.11). Following Sutradhar (2004), we may now write the GQL estimating equation for γ as

$$\sum_{i=1}^{K} \frac{\partial \lambda_i'}{\partial \gamma} M_i^{-1}(f_i - \lambda_i) = 0, \tag{2.49}$$

where $M_i = \text{cov}(f_i)$. Note that the derivatives of $\sigma_{it,t+1}$ with respect to γ are given in Appendix A. Thus, $\partial\lambda_i'/\partial\gamma$ in (2.49) is known. One may now solve (2.49) for γ, provided that the covariance matrix of f_i, i.e., M_i is known. Note, however, that as the distribution of $y_i = (y_{i1},\ldots,y_{it},\ldots,y_{iT})'$ may not be known, it is then impossible to derive the true covariance matrix of f_i. To solve this problem, similar to the estimation of γ by the GMM approach discussed in the last section, we, for convenience, pretend that y_i is normal with correct mean vector and variance matrix, that is,

$$y_i \sim N(\mu_i, \Sigma_i), \tag{2.50}$$

with μ_i and Σ_i as in (2.45). We then compute M_i for (2.49) under this normality assumption (2.50). Note that this 'working' normality assumption is used only for the construction of the weight matrix M_i in (2.49), the distance function $f_i - \lambda_i$ being the same as before as in (2.49) which was constructed without any distributional assumption for the data. Thus we solve the GQL estimating equation

$$\sum_{i=1}^{K} \frac{\partial \lambda_i'}{\partial \gamma} M_{iN}^{-1}(f_i - \lambda_i) = 0 \tag{2.51}$$

as a remedy to the use of the estimating equation (2.49). It is now clear that the elements of M_{iN} may be computed by using the general formula for the fourth order moments given by

$$E[(Y_{i\ell} - \mu_{i\ell})(Y_{im} - \mu_{im})(Y_{iu} - \mu_{iu})(Y_{it} - \mu_{it})] = \sigma_{i\ell m}\sigma_{iut} + \sigma_{i\ell u}\sigma_{imt} + \sigma_{imu}\sigma_{i\ell t}, \quad (2.52)$$

where σ_{itt} and σ_{iut} are elements of the true variance and covariance matrix Σ_i, their formulas are being given by (2.10) and (2.11), respectively.

Let $\hat{\gamma}_{GQL}$ be the solution of (2.51). It then follows that asymptotically ($K \to \infty$), $\hat{\gamma}_{GQL}$ has the univarite normal distribution given by

$$K^{\frac{1}{2}}(\hat{\gamma}_{GQL} - \gamma) \sim N(0, KV_1), \quad (2.53)$$

where V_1 has the formula

$$V_1 = \frac{\left[\sum_{i=1}^{K} \frac{\partial \lambda_i'}{\partial \gamma} M_{iN}^{-1} M_i M_{iN}^{-1} \frac{\partial \lambda_i}{\partial \gamma}\right]}{\left[\sum_{i=1}^{K} \frac{\partial \lambda_i'}{\partial \gamma} M_{iN}^{-1} \frac{\partial \lambda_i}{\partial \gamma}\right]^2}. \quad (2.54)$$

If the true distributions of the errors and random effects in model (1.4) were normal, the asymptotic covariance matrix V_1 in (2.54) would reduce to

$$V_1 = \frac{1}{\sum_{i=1}^{K} \frac{\partial \lambda_i'}{\partial \gamma} M_{iN}^{-1} \frac{\partial \lambda_i}{\partial \gamma}}. \quad (2.55)$$

Note that one can also use the raw second order responses to construct an alternative (to (2.48)) sufficient statistic given as

$$f_i^* = [y_{i1}y_{i2}, \ldots, y_{it}y_{it,t+1}, \ldots, y_{iT-1}y_{iT}], \quad (2.56)$$

The construction of the estimating equation based on f_i and f_i^* will however be similar. In the next chapter, we will also use the alternative estimating equation to be constructed based on f_i^* and study the properties of the estimators obtained from estimating equations based on both f_i and f_i^*.

GQL Estimation of $\alpha = (\sigma_\eta^2, \sigma_\epsilon^2)'$

Under the present model (1.4), σ_η^2 and σ_ϵ^2 are the variances of the random effects and error components, respectively. Note that by (2.10) and (2.11), $\alpha = (\sigma_\eta^2, \sigma_\epsilon^2)'$ is seen to be involved in the variances and covariances of the responses. Consequently, we write an elementary sufficient statistic vector consisting of the squares and pair-wise products of the responses, given by

$$\begin{aligned} s_i &= [(y_{i1} - \mu_{i1})^2, \ldots, (y_{it} - \mu_{it})^2, \ldots, (y_{iT} - \mu_{iT})^2, \\ &\quad (y_{i1} - \mu_{i1})(y_{i2} - \mu_{i2}), \ldots, (y_{iu} - \mu_{iu})(y_{it} - \mu_{it}), \\ &\quad \ldots, (y_{i,T-1} - \mu_{i,T-1})(y_{iT} - \mu_{iT})]', \end{aligned} \tag{2.57}$$

assuming that μ_{it}'s are known. The expectation of s_i is given by $\sigma_i = E(s_i) = (\sigma_{i11}, \ldots, \sigma_{itt}, \ldots, \sigma_{iTT}, \sigma_{i12}, \ldots, \sigma_{iut}, \ldots, \sigma_{i,T-1,T})'$, where $\sigma_{itt}, \sigma_{iut}$ are given by (2.10) and (2.11), respectively.

Similar to the GQL estimation of γ, we use the normal y_i based weight matrix $\Omega_{iN} = cov(s_i)$, and solve the GQL estimating equation for $\alpha = (\sigma_\eta^2, \sigma_\epsilon^2)'$ given by

$$\sum_{i=1}^{K} \frac{\partial \sigma_i'}{\partial \alpha} \Omega_{iN}^{-1} (s_i - \sigma_i) = 0. \tag{2.58}$$

Let $\hat{\alpha}_{GQL} = (\hat{\sigma}_{\eta,GQL}^2, \hat{\sigma}_{\epsilon,GQL}^2)'$ be the solution of (2.58). It then follows that asymptotically ($K \to \infty$)

$$K^{\frac{1}{2}} (\hat{\alpha}_{GQL} - \alpha) \sim N(0, KV_2), \tag{2.59}$$

where V_2 is given by

$$\begin{aligned} V_2 &= \left[\sum_{i=1}^{K} \frac{\partial \sigma_i'}{\partial \alpha} \Omega_{iN}^{-1} \frac{\partial \sigma_i}{\partial \alpha'}\right]^{-1} \left[\sum_{i=1}^{K} \frac{\partial \sigma_i'}{\partial \alpha} \Omega_{iN}^{-1} \Omega_i \Omega_{iN}^{-1} \frac{\partial \sigma_i}{\partial \alpha'}\right] \\ &\quad \times \left[\sum_{i=1}^{K} \frac{\partial \sigma_i'}{\partial \alpha} \Omega_{iN}^{-1} \frac{\partial \sigma_i}{\partial \alpha'}\right]^{-1}, \end{aligned} \tag{2.60}$$

with Ω_i as the true variance of s_i. If the true distributions of the errors and random effects in model (1.4) were normal, the asymptotic covariance matrix V_2 in (2.60) would reduce to

$$V_2 = \left[\sum_{i=1}^{K} \frac{\partial \sigma_i'}{\partial \alpha} \Omega_{iN}^{-1} \frac{\partial \sigma_i}{\partial \alpha'} \right]^{-1}. \quad (2.61)$$

Note that similar to the estimation of γ, one may like to use an alternative estimating equation for α, based on a slightly different sufficient statistic. To be specific, a vector of raw second order responses s_i^* given by

$$s_i^* = [y_{i1}^2, \ldots, y_{iT}^2, y_{i1}y_{i2}, \ldots, y_{iu}y_{it}, \ldots, y_{iT-1}y_{iT}], \quad (2.62)$$

can be used for such construction. In the next chapter, we use this approach as well for studying the properties of the estimators. Note however that using the estimating equation based on corrected second order responses is much simpler than using the estimating equation based on raw second order responses.

As far as the derivatives used in this chapter for different GMM and GQL estimating equations are concerned, we provide them in Appendix A. The computational formulas for C_1, C_{2N} and C_{3N} under the GMM approach, and for M_{iN} and Ω_{iN} under the GQL approach, are given in Appendix B.

Chapter 3

Asymptotic Efficiency Comparison

3.1 GMM versus GQL (Based on f_i and s_i)

To examine the asymptotic efficiency of the GQL approach over the GMM approach or vice versa, we need to compute the variances of the estimators under these two approaches. Note that under the assumption of normal distributions, for example, for the errors and random effects involved in the model (1.4), these variances can be computed from the formulas for the covariance matrices for the estimators under both GMM and GQL approaches given in Chapter 2. To be specific, the variances of the components of $\hat{\beta}_{GMM} = (\hat{\beta}_{1,GMM}, \hat{\beta}_{2,GMM})'$ and $\hat{\beta}_{GQL} = (\hat{\beta}_{1,GQL}, \hat{\beta}_{2,GQL})'$ can be found from the leading diagonal of the asymptotic $\mathrm{cov}(\hat{\beta}_{GMM})$ in (2.28) and $\mathrm{cov}(\hat{\beta}_{GQL})$ in (2.47), respectively. Similarly, the asymptotic variance of $\hat{\gamma}_{GMM}$ and $\hat{\gamma}_{GQL}$ are given directly by $\mathrm{var}(\hat{\gamma}_{GMM})$ in (2.37) and $\mathrm{var}(\hat{\gamma}_{GQL})$ in (2.55), respectively. Next, the asymptotic variances of the components of $\hat{\alpha}_{GMM} = (\hat{\sigma}^2_{\eta,GMM}, \hat{\sigma}^2_{\epsilon,GMM})'$ and $\hat{\alpha}_{GQL} = (\hat{\sigma}^2_{\eta,GQL}, \hat{\sigma}^2_{\epsilon,GQL})'$ are found from the leading diagonal of the $\mathrm{var}(\hat{\alpha}_{GMM})$ in (2.44) and $\mathrm{var}(\hat{\alpha}_{GQL})$ in (2.61), respectively.

In order to have a quantitative idea on the efficiency comparison, we have done an empirical study and computed the asymptotic variances under both GMM and GQL approaches, as described above. As far as the parameters of the model are concerned, we have chosen

$$\begin{aligned}
&K = 1000, p = 2, T = 6;\\
&\beta_1 = 0.5 \text{ or } 0.1, \beta_2 = 0.5; \gamma = \pm 0.8, \pm 0.3, 0.01;\\
&\sigma_\eta^2 = 0.5 \text{ or } 1.0, \text{ and } \sigma_\epsilon^2 = 1.0,
\end{aligned}$$

and for the $p = 2-$dimensional covariates for 1000 individuals over a period of 6 time points, we have chosen

$$x_{it1} = \begin{cases} 1 & \text{with } P(x_{it1} = 1) = 0.4 \\ 0 & \text{otherwise} \end{cases},$$

for all $i = 1, \ldots, 1000$ and $t = 1, \ldots, 6$; and

$$x_{it2} = \begin{cases} -1.0 & \text{for } i = 1, \ldots, K/4;\ t = 1, 2 \\ 0.0 & \text{for } i = 1, \ldots, K/4;\ t = 3, 4 \\ 1.0 & \text{for } i = 1, \ldots, K/4;\ t = 5, T \\ t/T & \text{for } i = K/4 + 1, \ldots, 3K/4;\ t = 1, T \\ 0.0 & \text{for } i = 3K/4 + 1, \ldots, K;\ t = 1, 3 \\ 1.0 & \text{for } i = 3K/4 + 1, \ldots, K;\ t = 4, T \end{cases}.$$

The asymptotic variances computed by using the above covariates and associated parameters are reported in Table 3.1 when $\beta_1 = \beta_2 = 0.5$ and in Table 3.2 when $\beta_1 = 0.1$ and $\beta_2 = 0.5$.

3.2 GMM versus GQL (Based on f_i^* and s_i^*)

Note that as opposed to the GMM approach, GQL approach is constructed based on suitable distance functions, for all individuals $i = 1, \ldots, K$. For the development of the GQL approach in Chapter 2, we have mainly used the basic statistics f_i (2.48) and s_i (2.57) to construct the necessary distance functions. Note however that these statistics are based on corrected second order responses by assuming that the parameters involved in them are known. While this approach appears to be reasonable, especially because the estimation is done by iteration, in this section, we provide the GQL estimating equations by avoiding parametric functions from the basic statistics. Thus, as mentioned in the last chapter, we consider the basic statistics f_i^* (2.56) and s_i^* (2.62) to construct the distance functions leading to the GQL estimating equations, especially for γ and $\alpha = (\sigma_\eta^2, \sigma_\epsilon^2)'$. We however caution that the computation for the formulas for necessary third and fourth order moments may be cumbersome as compared to those under the corrected response based GQL approach.

Since the GMM estimating equations for any parameters, as well as the GQL estimating equation for β do not require any direct use of basic statistics such as f_i^* and s_i^*, in this section we simply discuss the GQL estimating equations for γ and α to be constructed based on f_i^* and s_i^*. Note that we will refer to this f_i^* and s_i^* based GQL approach as GQL* approach. Similar to the f_i and s_i based GQL estimating equations for γ and α, the GQL* estimating equations for γ and α are given by

$$\sum_{i=1}^{K} \frac{\partial \lambda_i^{*\prime}}{\partial \gamma} M_{iN}^{*-1}(f_i^* - \lambda_i^*) = 0, \tag{3.1}$$

and

$$\sum_{i=1}^{K} \frac{\partial \sigma_i^{*\prime}}{\partial \alpha} \Omega_{iN}^{*-1}(s_i^* - \sigma_i^*) = 0, \tag{3.2}$$

respectively. In (3.1),

$$\begin{aligned}
\lambda_i^* = E(f_i^*) &= (\lambda_{i12}, \cdots, \lambda_{it,t+1}, \cdots, \lambda_{i,T-1,T})' \\
&= (\sigma_{i12} + \mu_{i1}\mu_{i2}, \cdots, \sigma_{it,t+1} + \mu_{it}\mu_{i,t+1}, \cdots, \sigma_{i,T-1,T} + \mu_{i,T-1}\mu_{iT})',
\end{aligned}$$

where μ_{it} and σ_{iut} are as given in (2.9) and (2.11), respectively, and M_{iN}^* is the pretended normal y_i based cov(f_i^*). Similarly, in (3.2),

$$\begin{aligned}
\sigma_i^* = E(s_i^*) &= (\lambda_{i11}, \cdots, \lambda_{iTT}, \lambda_{i12}, \cdots, \lambda_{iut}, \cdots, \lambda_{i,T-1,T})' \\
&= (\sigma_{i11} + \mu_{i1}^2, \cdots, \sigma_{iTT} + \mu_{iT}^2, \\
&\quad \sigma_{i12} + \mu_{i1}\mu_{i2}, \cdots, \sigma_{iut} + \mu_{iu}\mu_{it}, \cdots, \sigma_{i,T-1,T} + \mu_{i,T-1}\mu_{iT})',
\end{aligned}$$

and Ω^*_{iN} is the pretended normal y_i based $\mathrm{cov}(s^*_i)$.

Let $\hat{\gamma}_{GQL^*}$ and $\hat{\alpha}_{GQL^*}$ be the solutions to (3.1) and (3.2), respectively. Now, by using a similar approach for the computation of the variances of $\hat{\gamma}$ and $\hat{\alpha}$ under the GQL approach, the variances of $\hat{\gamma}_{GQL^*}$ and $\hat{\alpha}_{GQL^*}$ are given by

$$\mathrm{cov}(\hat{\gamma}_{GQL^*}) = \frac{1}{\sum_{i=1}^{K} \frac{\partial \lambda_i^{*'}}{\partial \gamma} M_{iN}^{*-1} \frac{\partial \lambda_i^*}{\partial \gamma}}, \tag{3.3}$$

and

$$\mathrm{cov}(\hat{\alpha}_{GQL^*}) = \left[\sum_{i=1}^{K} \frac{\partial \sigma_i^{*'}}{\partial \alpha} \Omega_{iN}^{*-1} \frac{\partial \sigma_i^*}{\partial \alpha'}\right]^{-1}, \tag{3.4}$$

respectively. The derivatives needed for (3.3) and (3.4) are found in Appendix A, and the computational formulas for M^*_{iN} and Ω^*_{iN} are found in Appendix B.

It is clear from Tables 3.1 and 3.2 that $\mathrm{var}(\hat{\beta}_u)$, for $u = 1, 2$, under the GQL approach are uniformly smaller than the corresponding variances under the GMM approach. In some occasions, the GMM approach may perform very poorly producing estimates with very low efficiency. For example, when $\beta_1 = \beta_2 = 0.5$, $\gamma = 0.8$, $\sigma^2_\eta = 0.5$, and $\sigma^2_\epsilon = 1.0$, the GQL estimates of β_1 and β_2 are respectively $\frac{0.315}{6.88\times 10.0^{-4}} = 457.85$ and $\frac{0.139}{5.19\times 10.0^{-4}} = 267.82$ times more efficient than the corresponding GMM estimates. Also, it is clear from Tables 3.1 and 3.2 that $\mathrm{var}(\hat{\sigma}^2_\eta)$ and $\mathrm{var}(\hat{\sigma}^2_\epsilon)$ under the GQL approach are uniformly smaller than the corresponding variances under the GMM approach. For example, for the above selected parameter values, the GQL estimate of σ^2_η is $\frac{1.13\times 10.0^{-3}}{8.89\times 10.0^{-4}} = 1.27$ times more efficient than the corresponding GMM estimate; and $\hat{\sigma}^2_{\epsilon,GQL}$ is $\frac{2.40\times 10.0^{-3}}{3.95\times 10.0^{-4}} = 6.08$ times more efficient than the corresponding $\hat{\sigma}^2_{\epsilon,GMM}$.

As far as the estimation of the dynamic dependence parameter γ is concerned, the GQL approach, in general, appears to produce estimators with smaller variances than the GMM approach. To be specific, only in a few cases GMM appears to produce estimator for γ, with smaller variance than the GQL approach. For example, when $\beta_1 = \beta_2 = 0.5$, $\gamma = -0.8$, $\sigma^2_\eta = 0.5$, and $\sigma^2_\epsilon = 1.0$, the GMM estimate of γ is $\frac{1.02\times 10.0^{-4}}{8.46\times 10.0^{-5}} = 1.21$ times more efficient than the corresponding GQL estimate. But in most of the cases, the GQL performs better than the GMM approach. For example, when $\beta_1 = \beta_2 = 0.5$, $\gamma = 0.01$, $\sigma^2_\eta = 1.0$, and $\sigma^2_\epsilon = 1.0$, the GQL estimate of γ is $\frac{5.07\times10.0^{-4}}{1.75\times10.0^{-4}} = 2.90$ times more efficient than the corresponding GMM estimate.

Note that in some cases, the GMM and GQL approaches produce negative variances, with the GMM approach being worse. These cases are indicated by '-'. This happens when the weight matrix in the estimating equations is not positive definite due to the selection of the values for the covariates. But, it does not appear to be a serious problem as in most of the cases, the variances are found to be positive.

Next, the variances of the GQL* estimators are also reported in the same Tables 3.1 and 3.2. As mentioned before, the purpose of using the GQL* approach is to examine whether the raw responses based estimation works better than the corrected responses based GQL approach, ever though GQL* approach is computationally more cumbersome. It is however found from the results in Tables 3.1 and 3.2 that these approaches are highly competitive to each other. This is because, (1) GQL and GQL* are the same in theory for the estimation of β; (2) the GQL* appears to be uniformly better than the GQL approach in estimating the dynamic dependence parameter γ; (3) but GQL* performs worse than the GQL approach in estimating σ^2_η; and (4) for the remaining σ^2_ϵ parameter, both approaches appear to be competitive, GQL being slightly better. Note that in summary, we, however, recommend the use of the GQL approach as compared to the GQL* approach because of its computational simplicity.

Table 3.1: Asymptotic variances of the GMM and GQL estimators for the parameters of the dynamic model (1.4) for $\sigma_\epsilon^2 = 1.0$ and selected true values of γ and σ_η^2 when $\beta_1 = \beta_2 = 0.5$.

			Asymptotic variances				
σ_η^2	γ	Method	Var($\hat\beta_1$)	Var($\hat\beta_2$)	Var($\hat\gamma$)	Var($\hat\sigma_\eta^2$)	Var($\hat\sigma_\epsilon^2$)
0.5	-0.8	GQL	6.88×10.0^{-4}	5.19×10.0^{-4}	1.02×10.0^{-4}	8.89×10.0^{-4}	3.95×10.0^{-4}
		GQL*	6.88×10.0^{-4}	5.19×10.0^{-4}	8.28×10.0^{-5}	1.29×10.0^{-3}	-
		GMM	0.234	-	8.46×10.0^{-5}	1.05×10.0^{-3}	1.04×10.0^{-3}
	-0.3	GQL	6.88×10.0^{-4}	5.19×10.0^{-4}	1.78×10.0^{-4}	8.89×10.0^{-4}	3.95×10.0^{-4}
		GQL*	6.88×10.0^{-4}	5.19×10.0^{-4}	9.99×10.0^{-5}	1.19×10.0^{-3}	4.29×10.0^{-4}
		GMM	6.54×10.0^{-3}	-	2.17×10.0^{-4}	9.08×10.0^{-4}	4.57×10.0^{-4}
	0.01	GQL	6.88×10.0^{-4}	5.19×10.0^{-4}	1.88×10.0^{-4}	8.89×10.0^{-4}	3.95×10.0^{-4}
		GQL*	6.88×10.0^{-4}	5.19×10.0^{-4}	7.02×10.0^{-5}	1.20×10.0^{-3}	4.28×10.0^{-4}
		GMM	3.81×10.0^{-3}	7.53×10.0^{-3}	3.29×10.0^{-4}	9.00×10.0^{-4}	4.00×10.0^{-4}
	0.3	GQL	6.88×10.0^{-4}	5.19×10.0^{-4}	1.68×10.0^{-4}	8.89×10.0^{-4}	3.95×10.0^{-4}
		GQL*	6.88×10.0^{-4}	5.19×10.0^{-4}	2.84×10.0^{-5}	1.20×10.0^{-3}	3.93×10.0^{-4}
		GMM	0.142	0.153	2.97×10.0^{-4}	9.09×10.0^{-4}	4.60×10.0^{-4}
	0.8	GQL	6.88×10.0^{-4}	5.19×10.0^{-4}	8.02×10.0^{-5}	8.89×10.0^{-4}	3.95×10.0^{-4}
		GQL*	6.88×10.0^{-4}	5.19×10.0^{-4}	1.27×10.0^{-6}	1.20×10.0^{-3}	8.99×10.0^{-4}
		GMM	0.315	0.139	9.14×10.0^{-5}	1.13×10.0^{-3}	2.40×10.0^{-3}
1.0	-0.8	GQL	7.43×10.0^{-4}	5.55×10.0^{-4}	9.84×10.0^{-5}	2.72×10.0^{-3}	3.98×10.0^{-4}
		GQL*	7.43×10.0^{-4}	5.55×10.0^{-4}	3.11×10.0^{-5}	3.28×10.0^{-3}	3.76×10.0^{-4}
		GMM	0.251	-	8.63×10.0^{-5}	2.99×10.0^{-3}	1.09×10.0^{-3}
	-0.3	GQL	7.43×10.0^{-4}	5.55×10.0^{-4}	1.70×10.0^{-4}	2.72×10.0^{-3}	3.98×10.0^{-4}
		GQL*	7.43×10.0^{-4}	5.55×10.0^{-4}	5.86×10.0^{-5}	3.27×10.0^{-3}	4.30×10.0^{-4}
		GMM	7.26×10.0^{-3}	-	3.44×10.0^{-4}	2.75×10.0^{-3}	4.61×10.0^{-4}
	0.01	GQL	7.43×10.0^{-4}	5.55×10.0^{-4}	1.75×10.0^{-4}	2.72×10.0^{-3}	3.98×10.0^{-4}
		GQL*	7.43×10.0^{-4}	5.55×10.0^{-4}	3.17×10.0^{-5}	3.27×10.0^{-3}	4.22×10.0^{-4}
		GMM	4.28×10.0^{-3}	7.86×10.0^{-3}	5.07×10.0^{-4}	2.73×10.0^{-3}	4.00×10.0^{-4}
	0.3	GQL	7.43×10.0^{-4}	5.55×10.0^{-4}	1.49×10.0^{-4}	2.72×10.0^{-3}	3.98×10.0^{-4}
		GQL*	7.43×10.0^{-4}	5.55×10.0^{-4}	1.01×10.0^{-5}	3.27×10.0^{-3}	3.95×10.0^{-4}
		GMM	0.134	0.144	3.66×10.0^{-4}	2.75×10.0^{-3}	4.75×10.0^{-4}
	0.8	GQL	7.43×10.0^{-4}	5.55×10.0^{-4}	6.13×10.0^{-5}	2.72×10.0^{-3}	3.98×10.0^{-4}
		GQL*	7.43×10.0^{-4}	5.55×10.0^{-4}	3.23×10.0^{-7}	3.24×10.0^{-3}	4.09×10.0^{-5}
		GMM	0.300	0.133	9.24×10.0^{-5}	3.12×10.0^{-3}	3.77×10.0^{-3}

Table 3.2: Asymptotic variances of the GMM and GQL estimators for the parameters of the dynamic model (1.4) for $\sigma_\epsilon^2 = 1.0$ and selected true values of γ and σ_η^2 when $\beta_1 = 0.1$ and $\beta_2 = 0.5$.

			Asymptotic variances				
σ_η^2	γ	Method	Var($\hat\beta_1$)	Var($\hat\beta_2$)	Var($\hat\gamma$)	Var($\hat\sigma_\eta^2$)	Var($\hat\sigma_\epsilon^2$)
0.5	-0.8	GQL	6.88×10.0^{-4}	5.19×10.0^{-4}	1.02×10.0^{-4}	8.89×10.0^{-4}	3.95×10.0^{-4}
		GQL*	6.88×10.0^{-4}	5.19×10.0^{-4}	7.26×10.0^{-5}	1.03×10.0^{-3}	4.05×10.0^{-4}
		GMM	8.29×10.0^{-3}	-	8.54×10.0^{-5}	1.05×10.0^{-3}	1.04×10.0^{-3}
	-0.3	GQL	6.88×10.0^{-4}	5.19×10.0^{-4}	1.78×10.0^{-4}	8.89×10.0^{-4}	3.95×10.0^{-4}
		GQL*	6.88×10.0^{-4}	5.19×10.0^{-4}	1.01×10.0^{-4}	1.05×10.0^{-3}	4.23×10.0^{-4}
		GMM	-	-	2.14×10.0^{-4}	9.08×10.0^{-4}	4.57×10.0^{-4}
	0.01	GQL	6.88×10.0^{-4}	5.19×10.0^{-4}	1.88×10.0^{-4}	8.89×10.0^{-4}	3.95×10.0^{-4}
		GQL*	6.88×10.0^{-4}	5.19×10.0^{-4}	6.96×10.0^{-5}	1.05×10.0^{-3}	4.24×10.0^{-4}
		GMM	1.22×10.0^{-3}	6.22×10.0^{-3}	3.29×10.0^{-4}	9.00×10.0^{-4}	4.00×10.0^{-4}
	0.3	GQL	6.88×10.0^{-4}	5.19×10.0^{-4}	1.68×10.0^{-4}	8.89×10.0^{-4}	3.95×10.0^{-4}
		GQL*	6.88×10.0^{-4}	5.19×10.0^{-4}	2.79×10.0^{-5}	1.05×10.0^{-3}	4.17×10.0^{-4}
		GMM	1.19×10.0^{-2}	0.132	2.96×10.0^{-4}	9.09×10.0^{-4}	4.60×10.0^{-4}
	0.8	GQL	6.88×10.0^{-4}	5.19×10.0^{-4}	8.02×10.0^{-5}	8.89×10.0^{-4}	3.95×10.0^{-4}
		GQL*	6.88×10.0^{-4}	5.19×10.0^{-4}	1.23×10.0^{-6}	1.05×10.0^{-3}	4.62×10.0^{-4}
		GMM	2.11×10.0^{-2}	0.162	9.08×10.0^{-5}	1.13×10.0^{-3}	2.40×10.0^{-3}
1.0	-0.8	GQL	7.43×10.0^{-4}	5.55×10.0^{-4}	9.84×10.0^{-5}	2.72×10.0^{-3}	3.98×10.0^{-4}
		GQL*	7.43×10.0^{-4}	5.55×10.0^{-4}	4.73×10.0^{-5}	2.99×10.0^{-3}	3.87×10.0^{-4}
		GMM	8.86×10.0^{-3}	-	8.60×10.0^{-5}	2.99×10.0^{-3}	1.09×10.0^{-3}
	-0.3	GQL	7.43×10.0^{-4}	5.55×10.0^{-4}	1.70×10.0^{-4}	2.72×10.0^{-3}	3.98×10.0^{-4}
		GQL*	7.43×10.0^{-4}	5.55×10.0^{-4}	5.78×10.0^{-5}	3.01×10.0^{-3}	4.27×10.0^{-4}
		GMM	-	-	3.41×10.0^{-4}	2.75×10.0^{-3}	4.61×10.0^{-4}
	0.01	GQL	7.43×10.0^{-4}	5.55×10.0^{-4}	1.75×10.0^{-4}	2.72×10.0^{-3}	3.98×10.0^{-4}
		GQL*	7.43×10.0^{-4}	5.55×10.0^{-4}	3.08×10.0^{-5}	3.01×10.0^{-3}	4.28×10.0^{-4}
		GMM	1.72×10.0^{-3}	6.53×10.0^{-3}	5.07×10.0^{-4}	2.73×10.0^{-3}	4.00×10.0^{-4}
	0.3	GQL	7.43×10.0^{-4}	5.55×10.0^{-4}	1.49×10.0^{-4}	2.72×10.0^{-3}	3.98×10.0^{-4}
		GQL*	7.43×10.0^{-4}	5.55×10.0^{-4}	9.30×10.0^{-6}	3.01×10.0^{-3}	4.23×10.0^{-4}
		GMM	1.22×10.0^{-2}	0.124	3.66×10.0^{-4}	2.75×10.0^{-3}	4.75×10.0^{-4}
	0.8	GQL	7.43×10.0^{-4}	5.55×10.0^{-4}	6.13×10.0^{-5}	2.72×10.0^{-3}	3.98×10.0^{-4}
		GQL*	7.43×10.0^{-4}	5.55×10.0^{-4}	3.24×10.0^{-7}	3.01×10.0^{-3}	4.38×10.0^{-4}
		GMM	2.24×10.0^{-2}	0.154	9.23×10.0^{-5}	3.12×10.0^{-3}	3.77×10.0^{-3}

Chapter 4

Relative Performances of the GQL and GMM Approaches: A Simulation Study

In this chapter, we construct a simulation study to examine the small sample behavior of the GMM and GQL estimating equations discussed in section 2.2, for the estimation of the parameters β, γ, σ_η^2 and σ_ϵ^2 involved in the model (1.4). For this purpose, we first generate data $y_{i1}, \ldots, y_{it}, \ldots, y_{iT}$ for $i = 1, \ldots, K$ and $t = 1, \ldots, T$, by (1.4) in each of 500 or more simulations, using the same design parameters and covariates as in Chapter 3 (see section 3.1 for details) except that we now consider total number of individuals as $K = 100$, whereas in the asymptotic variance study in Chapter 3, K was considered to be $K = 1000$.

4.1 Estimation of β When Other Parameters Are Known

Recall from section 2.2 that the GMM and GQL estimates of β were obtained by using the iterative equations (2.26) and (2.46), respectively. We apply these equations to the generated data in a given simulation and obtain the GMM and GQL estimates for β. The simulation is repeated for 1000 times. We then compute the simulated mean (SM) and the simulated standard error (SSE) of these 1000 estimates for each of the two components β_1 and β_2. These simulated means and standard errors are reported in Tables 4.1 and 4.2 for both GMM and GQL approaches. We also computed the simulated mean squared errors (SMSE) by using, for example,

$$SMSE(\hat{\beta}_{1,GMM}) = (\bar{\hat{\beta}}_{1,GMM} - \beta_1)^2 + SSE^2(\hat{\beta}_{1,GMM}), \tag{4.1}$$

where

$$\bar{\hat{\beta}}_{1,GMM} = \frac{\sum_{s=1}^{1000} \hat{\beta}_{1,GMM}^{(s)}}{1000}, \tag{4.2}$$

and

$$SSE^2(\hat{\beta}_{1,GMM}) = \frac{\sum_{s=1}^{1000} (\hat{\beta}_{1,GMM}^{(s)} - \bar{\hat{\beta}}_{1,GMM})^2}{1000}. \tag{4.3}$$

The computed SMSE's are reported in the same Tables 4.1 and 4.2. We have also computed and reported the estimated standard errors (ESE) for both of $\hat{\beta}_1$ and $\hat{\beta}_2$. For the computation of the ESE under the GMM approach, for example, we evaluate the $\text{cov}(\hat{\beta}_{GMM}) = \left\{\frac{\partial \psi_1'}{\partial \beta} C_1 \frac{\partial \psi_1}{\partial \beta'}\right\}^{-1}$ by (2.28) in a given simulation by using $\beta = \hat{\beta}_{GMM}$. We then take the average of 1000 simulated diagonal elements of this estimated covariance matrix. These averages represent the ESE of $\hat{\beta}_{1,GMM}$ and $\hat{\beta}_{2,GMM}$. In a similar way, we obtain the ESE for the GQL estimates by using the covariate matrix $\text{cov}(\hat{\beta}_{GQL}) = \left[\sum_{i=1}^{K} \frac{\partial \mu_i'}{\partial \beta} \Sigma_i^{-1} \frac{\partial \mu_i}{\partial \beta'}\right]^{-1}$ as in (2.47).

It is clear from Tables 4.1 and 4.2 that the GQL approach uniformly outperforms the GMM approach in the estimation of β_u, for $u = 1, 2$. To be specific, GQL produces estimates of β_1 and β_2, with smaller biases, SSE and SMSE, than those of the GMM approach. For example, for the estimation of $\beta_1 = \beta_2 = 0.5$ when $\gamma = 0.8$, $\sigma_\eta^2 = 1.0$ and $\sigma_\epsilon^2 = 1.0$, GQL produces zero bias for β_1 as compared with the bias 0.039 by GMM, and for β_2, GQL has a smaller bias 0.002 as compared with the bias 0.013 by GMM. Furthermore, for β_1, the SSE under the GMM is $\frac{0.214}{8.54\times 10.0^{-2}} = 2.506$ times of the SSE under the GQL, and the SMSE under the GMM is $\frac{4.74\times 10.0^{-2}}{7.29\times 10.0^{-3}} = 6.502$ times of the SMSE under the GQL. Similarly, for β_2, the SSE under the GMM is

Table 4.1: The simulated GQL and GMM estimates for β_1 and β_2 under the dynamic model (1.4) with true values as $\beta_1 = \beta_2 = 0.5$; when γ, σ_η^2 and σ_ϵ^2 ($\sigma_\epsilon^2 = 1.0$) are known.

				Quantities			
σ_η^2	γ	Method		SM	SSE	ESE	$SMSE$
0.5	-0.8	GQL	β_1	0.500	8.26×10.0^{-2}	8.29×10.0^{-2}	6.83×10.0^{-3}
			β_2	0.501	7.11×10.0^{-2}	7.18×10.0^{-2}	5.06×10.0^{-3}
		GMM	β_1	0.494	0.125	0.137	1.57×10.0^{-2}
			β_2	0.475	0.131	0.138	1.77×10.0^{-2}
	-0.3	GQL	β_1	0.500	8.26×10.0^{-2}	8.29×10.0^{-2}	6.83×10.0^{-3}
			β_2	0.501	7.11×10.0^{-2}	7.18×10.0^{-2}	5.06×10.0^{-3}
		GMM	β_1	0.465	8.70×10.0^{-2}	9.53×10.0^{-2}	8.83×10.0^{-3}
			β_2	0.480	9.49×10.0^{-2}	9.87×10.0^{-2}	9.40×10.0^{-3}
	0.01	GQL	β_1	0.500	8.26×10.0^{-2}	8.29×10.0^{-2}	6.83×10.0^{-3}
			β_2	0.501	7.11×10.0^{-2}	7.18×10.0^{-2}	5.06×10.0^{-3}
		GMM	β_1	0.472	9.39×10.0^{-2}	0.105	9.60×10.0^{-3}
			β_2	0.483	9.56×10.0^{-2}	9.89×10.0^{-2}	9.42×10.0^{-3}
	0.3	GQL	β_1	0.500	8.26×10.0^{-2}	8.29×10.0^{-2}	6.83×10.0^{-3}
			β_2	0.501	7.11×10.0^{-2}	7.18×10.0^{-2}	5.06×10.0^{-3}
		GMM	β_1	0.473	0.113	0.129	1.34×10.0^{-2}
			β_2	0.489	0.102	0.107	1.05×10.0^{-2}
	0.8	GQL	β_1	0.500	8.26×10.0^{-2}	8.29×10.0^{-2}	6.83×10.0^{-3}
			β_2	0.501	7.11×10.0^{-2}	7.18×10.0^{-2}	5.06×10.0^{-3}
		GMM	β_1	0.462	0.165	0.200	2.87×10.0^{-2}
			β_2	0.514	0.152	0.174	2.32×10.0^{-2}
1.0	-0.8	GQL	β_1	0.500	8.54×10.0^{-2}	8.62×10.0^{-2}	7.29×10.0^{-3}
			β_2	0.502	7.37×10.0^{-2}	7.43×10.0^{-2}	5.43×10.0^{-3}
		GMM	β_1	0.494	0.128	0.140	1.64×10.0^{-2}
			β_2	0.475	0.141	0.148	2.04×10.0^{-2}
	-0.3	GQL	β_1	0.500	8.54×10.0^{-2}	8.62×10.0^{-2}	7.29×10.0^{-3}
			β_2	0.502	7.37×10.0^{-2}	7.43×10.0^{-2}	5.43×10.0^{-3}
		GMM	β_1	0.464	9.71×10.0^{-2}	0.107	1.07×10.0^{-2}
			β_2	0.479	0.115	0.119	1.37×10.0^{-2}
	0.01	GQL	β_1	0.500	8.54×10.0^{-2}	8.62×10.0^{-2}	7.29×10.0^{-3}
			β_2	0.502	7.37×10.0^{-2}	7.43×10.0^{-2}	5.43×10.0^{-3}
		GMM	β_1	0.471	0.113	0.128	1.35×10.0^{-2}
			β_2	0.483	0.118	0.121	1.42×10.0^{-2}
	0.3	GQL	β_1	0.500	8.54×10.0^{-2}	8.62×10.0^{-2}	7.29×10.0^{-3}
			β_2	0.502	7.37×10.0^{-2}	7.43×10.0^{-2}	5.43×10.0^{-3}
		GMM	β_1	0.473	0.141	0.164	2.06×10.0^{-2}
			β_2	0.488	0.127	0.133	1.63×10.0^{-2}
	0.8	GQL	β_1	0.500	8.54×10.0^{-2}	8.62×10.0^{-2}	7.29×10.0^{-3}
			β_2	0.502	7.37×10.0^{-2}	7.43×10.0^{-2}	5.43×10.0^{-3}
		GMM	β_1	0.461	0.214	0.262	4.74×10.0^{-2}
			β_2	0.513	0.198	0.226	3.94×10.0^{-2}

Table 4.2: The simulated GQL and GMM estimates for β_1 and β_2 under the dynamic model (1.4) with true values as $\beta_1 = 0.1$, $\beta_2 = 0.5$; when γ, σ_η^2 and σ_ϵ^2 ($\sigma_\epsilon^2 = 1.0$) are known.

				Quantities			
σ_η^2	γ	Method		SM	SSE	ESE	$SMSE$
0.5	-0.8	GQL	β_1	0.100	8.26×10.0^{-2}	8.29×10.0^{-2}	6.83×10.0^{-3}
			β_2	0.501	7.11×10.0^{-2}	7.18×10.0^{-2}	5.06×10.0^{-3}
		GMM	β_1	9.10×10.0^{-2}	0.123	0.137	1.58×10.0^{-2}
			β_2	0.479	0.131	0.139	1.77×10.0^{-2}
	-0.3	GQL	β_1	0.100	8.26×10.0^{-2}	8.29×10.0^{-2}	6.83×10.0^{-3}
			β_2	0.501	7.11×10.0^{-2}	7.18×10.0^{-2}	5.06×10.0^{-3}
		GMM	β_1	9.84×10.0^{-2}	8.70×10.0^{-2}	9.53×10.0^{-2}	7.57×10.0^{-3}
			β_2	0.474	9.49×10.0^{-2}	9.87×10.0^{-2}	9.69×10.0^{-3}
	0.01	GQL	β_1	0.100	8.26×10.0^{-2}	8.29×10.0^{-2}	6.83×10.0^{-3}
			β_2	0.501	7.11×10.0^{-2}	7.18×10.0^{-2}	5.06×10.0^{-3}
		GMM	β_1	0.105	9.39×10.0^{-2}	0.105	8.83×10.0^{-3}
			β_2	0.474	9.56×10.0^{-2}	9.89×10.0^{-2}	9.80×10.0^{-3}
	0.3	GQL	β_1	0.100	8.26×10.0^{-2}	8.29×10.0^{-2}	6.83×10.0^{-3}
			β_2	0.501	7.11×10.0^{-2}	7.18×10.0^{-2}	5.06×10.0^{-3}
		GMM	β_1	0.109	0.113	0.129	1.28×10.0^{-2}
			β_2	0.475	0.102	0.107	1.09×10.0^{-2}
	0.8	GQL	β_1	0.100	8.26×10.0^{-2}	8.29×10.0^{-2}	6.83×10.0^{-3}
			β_2	0.501	7.11×10.0^{-2}	7.18×10.0^{-2}	5.06×10.0^{-3}
		GMM	β_1	0.109	0.165	0.200	2.73×10.0^{-2}
			β_2	0.482	0.152	0.174	2.33×10.0^{-2}
1.0	-0.8	GQL	β_1	0.100	8.54×10.0^{-2}	8.62×10.0^{-2}	7.29×10.0^{-3}
			β_2	0.502	7.37×10.0^{-2}	7.43×10.0^{-2}	5.43×10.0^{-3}
		GMM	β_1	9.08×10.0^{-2}	0.128	0.140	1.65×10.0^{-2}
			β_2	0.478	0.141	0.148	2.04×10.0^{-2}
	-0.3	GQL	β_1	0.100	8.54×10.0^{-2}	8.62×10.0^{-2}	7.29×10.0^{-3}
			β_2	0.502	7.37×10.0^{-2}	7.43×10.0^{-2}	5.43×10.0^{-3}
		GMM	β_1	9.79×10.0^{-2}	9.71×10.0^{-2}	0.107	9.43×10.0^{-3}
			β_2	0.473	0.115	0.119	1.40×10.0^{-2}
	0.01	GQL	β_1	0.100	8.54×10.0^{-2}	8.62×10.0^{-2}	7.29×10.0^{-3}
			β_2	0.502	7.37×10.0^{-2}	7.43×10.0^{-2}	5.43×10.0^{-3}
		GMM	β_1	0.104	0.113	0.128	1.27×10.0^{-2}
			β_2	0.4744	0.118	0.121	1.45×10.0^{-2}
	0.3	GQL	β_1	0.100	8.54×10.0^{-2}	8.62×10.0^{-2}	7.29×10.0^{-3}
			β_2	0.502	7.37×10.0^{-2}	7.43×10.0^{-2}	5.43×10.0^{-3}
		GMM	β_1	0.109	0.141	0.164	1.99×10.0^{-2}
			β_2	0.475	0.127	0.133	1.68×10.0^{-2}
	0.8	GQL	β_1	0.100	8.54×10.0^{-2}	8.62×10.0^{-2}	7.29×10.0^{-3}
			β_2	0.502	7.37×10.0^{-2}	7.43×10.0^{-2}	5.43×10.0^{-3}
		GMM	β_1	0.107	0.214	0.262	4.60×10.0^{-2}
			β_2	0.481	0.198	0.226	3.96×10.0^{-2}

$\frac{0.198}{7.37\times 10.0^{-2}} = 2.687$ times of the SSE under the GQL, and the SMSE under the GMM approach is $\frac{3.94\times 10.0^{-2}}{5.43\times 10.0^{-3}} = 7.256$ times of the SMSE under the GQL.

As far as the performance of the ESE is concerned, ESEs appear to be close to the corresponding SSEs under both GMM and GQL approaches, the closeness under the GQL approach being much better than under the GMM approach. For example, for the above selected parameter values, the SSE($\hat{\beta}_{1,GQL}$) is 8.54×10.0^{-2} and the ESE($\hat{\beta}_{1,GQL}$) is 8.62×10.0^{-2}, which are very close to each other as compared to SSE($\hat{\beta}_{1,GMM}$) = 0.214 and ESE($\hat{\beta}_{1,GMM}$) = 0.262. Similar results hold for $\hat{\beta}_2$ under both GMM and GQL approaches.

4.2 Estimation of β and γ When σ_η^2 and σ_ϵ^2 Are Known

Given that the simulation results in section 4.1 show that the GMM and GQL approaches are performing well for β estimation, GQL being better than GMM, we now include one more parameter γ and estimate β and γ together under 1000 simulations, where σ_η^2 and σ_ϵ^2 are still assumed to be known.

For estimation of β and γ under the GMM approach, we use the iterative equation (2.26) for β, and (2.35) for γ. Similarly, for the GQL estimates, we use the iterative equation (2.46) for β, and the iterative equation for γ given by

$$\hat{\gamma}_{GQL,(r+1)} = \hat{\gamma}_{GQL,(r)} + \left[\sum_{i=1}^{K} \frac{\partial \lambda_i'}{\partial \gamma} M_{iN}^{-1} \frac{\partial \lambda_i}{\partial \gamma}\right]_{(r)}^{-1} \left[\sum_{i=1}^{K} \frac{\partial \lambda_i'}{\partial \gamma} M_{iN}^{-1} (f_i - \lambda_i)\right]_{(r)}, \tag{4.4}$$

where f_i is as given in (2.48), and λ_i, M_{iN} are given for the GQL estimation of γ in section 2.2.

The SM, SSE, ESE and SMSE for $\hat{\beta}$ and $\hat{\gamma}$ under both GQL and GMM are given in Tables 4.3-4.4. Note that even though we estimate β and γ, the performance of the GMM and GQL estimates for β remains the same as in section 4.1 where β was estimated assuming γ known. We now interpret the performance of these approaches for γ estimate. From these tables, we can see that in most cases, GQL

Table 4.3: The simulated GQL and GMM estimates for β and γ under the dynamic model (1.4) with true regression parameter values as $\beta_1 = \beta_2 = 0.5$ and selected true values for γ; when σ_η^2 and σ_ϵ^2 ($\sigma_\epsilon^2 = 1.0$) are known.

				Quantities			
σ_η^2	γ	Method		SM	SSE	ESE	$SMSE$
0.5	-0.8	GQL	β_1	0.500	8.28×10.0^{-2}	8.29×10.0^{-2}	6.86×10.0^{-3}
			β_2	0.501	7.21×10.0^{-2}	7.18×10.0^{-2}	5.20×10.0^{-3}
			γ	-0.798	3.96×10.0^{-2}	2.72×10.0^{-2}	1.58×10.0^{-3}
		GMM	β_1	0.495	0.135	0.129	1.83×10.0^{-2}
			β_2	0.505	0.114	0.118	1.30×10.0^{-2}
			γ	-0.793	4.01×10.0^{-2}	3.91×10.0^{-2}	1.66×10.0^{-3}
	-0.3	GQL	β_1	0.500	8.30×10.0^{-2}	8.29×10.0^{-2}	6.88×10.0^{-3}
			β_2	0.501	7.38×10.0^{-2}	7.18×10.0^{-2}	5.45×10.0^{-3}
			γ	-0.300	4.91×10.0^{-2}	5.29×10.0^{-2}	2.41×10.0^{-3}
		GMM	β_1	0.496	9.28×10.0^{-2}	9.05×10.0^{-2}	8.63×10.0^{-3}
			β_2	0.501	8.59×10.0^{-2}	8.44×10.0^{-2}	7.38×10.0^{-3}
			γ	-0.300	4.84×10.0^{-2}	4.72×10.0^{-2}	2.35×10.0^{-3}
	0.01	GQL	β_1	0.500	8.30×10.0^{-2}	8.29×10.0^{-2}	6.89×10.0^{-3}
			β_2	0.503	7.62×10.0^{-2}	7.18×10.0^{-2}	5.81×10.0^{-3}
			γ	6.99×10.0^{-3}	5.60×10.0^{-2}	6.85×10.0^{-2}	3.14×10.0^{-3}
		GMM	β_1	0.498	0.102	9.91×10.0^{-2}	1.03×10.0^{-2}
			β_2	0.503	9.05×10.0^{-2}	8.49×10.0^{-2}	8.21×10.0^{-3}
			γ	4.83×10.0^{-3}	5.55×10.0^{-2}	5.41×10.0^{-2}	3.11×10.0^{-3}
	0.3	GQL	β_1	0.500	8.31×10.0^{-2}	8.29×10.0^{-2}	6.90×10.0^{-3}
			β_2	0.504	7.84×10.0^{-2}	7.18×10.0^{-2}	6.16×10.0^{-3}
			γ	0.294	5.45×10.0^{-2}	7.05×10.0^{-2}	3.00×10.0^{-3}
		GMM	β_1	0.500	0.124	0.121	1.54×10.0^{-2}
			β_2	0.505	0.100	9.16×10.0^{-2}	1.01×10.0^{-2}
			γ	0.292	5.60×10.0^{-2}	5.43×10.0^{-2}	3.20×10.0^{-3}
	0.8	GQL	β_1	0.500	8.30×10.0^{-2}	8.29×10.0^{-2}	6.89×10.0^{-3}
			β_2	0.505	7.76×10.0^{-2}	7.18×10.0^{-2}	6.04×10.0^{-3}
			γ	0.796	3.34×10.0^{-2}	3.69×10.0^{-2}	1.14×10.0^{-3}
		GMM	β_1	0.500	0.183	0.177	3.35×10.0^{-2}
			β_2	0.508	0.155	0.146	2.42×10.0^{-2}
			γ	0.791	4.47×10.0^{-2}	4.31×10.0^{-2}	2.07×10.0^{-3}
1.0	-0.8	GQL	β_1	0.500	8.56×10.0^{-2}	8.62×10.0^{-2}	7.33×10.0^{-3}
			β_2	0.502	7.48×10.0^{-2}	7.43×10.0^{-2}	5.60×10.0^{-3}
			γ	-0.800	4.45×10.0^{-2}	2.04×10.0^{-2}	1.98×10.0^{-3}
		GMM	β_1	0.495	0.138	0.132	1.92×10.0^{-2}
			β_2	0.505	0.122	0.126	1.48×10.0^{-2}
			γ	-0.794	3.86×10.0^{-2}	3.76×10.0^{-2}	1.53×10.0^{-3}

(Table 4.3 contd....)

σ_η^2	γ	Method		Quantities			
				SM	SSE	ESE	$SMSE$
	-0.3	GQL	β_1	0.500	8.57×10.0^{-2}	8.62×10.0^{-2}	7.35×10.0^{-3}
			β_2	0.502	7.69×10.0^{-2}	7.43×10.0^{-2}	5.92×10.0^{-3}
			γ	-0.301	5.89×10.0^{-2}	6.23×10.0^{-2}	3.47×10.0^{-3}
		GMM	β_1	0.495	0.105	0.103	1.11×10.0^{-2}
			β_2	0.501	0.102	0.101	1.05×10.0^{-2}
			γ	-0.302	5.46×10.0^{-2}	5.29×10.0^{-2}	2.99×10.0^{-3}
	0.01	GQL	β_1	0.500	8.58×10.0^{-2}	8.62×10.0^{-2}	7.37×10.0^{-3}
			β_2	0.504	7.93×10.0^{-2}	7.43×10.0^{-2}	6.30×10.0^{-3}
			γ	4.76×10.0^{-3}	6.36×10.0^{-2}	8.46×10.0^{-2}	4.07×10.0^{-3}
		GMM	β_1	0.497	0.124	0.121	1.54×10.0^{-2}
			β_2	0.503	0.109	0.103	1.19×10.0^{-2}
			γ	2.41×10.0^{-3}	6.27×10.0^{-2}	6.08×10.0^{-2}	3.99×10.0^{-3}
	0.3	GQL	β_1	0.500	8.59×10.0^{-2}	8.62×10.0^{-2}	7.37×10.0^{-3}
			β_2	0.505	8.04×10.0^{-2}	7.43×10.0^{-2}	6.49×10.0^{-3}
			γ	0.294	5.49×10.0^{-2}	7.84×10.0^{-2}	3.05×10.0^{-3}
		GMM	β_1	0.499	0.157	0.154	2.46×10.0^{-2}
			β_2	0.505	0.121	0.113	1.48×10.0^{-2}
			γ	0.290	5.96×10.0^{-2}	5.76×10.0^{-2}	3.65×10.0^{-3}
	0.8	GQL	β_1	0.500	8.58×10.0^{-2}	8.62×10.0^{-2}	7.36×10.0^{-3}
			β_2	0.504	7.81×10.0^{-2}	7.43×10.0^{-2}	6.11×10.0^{-3}
			γ	0.797	2.85×10.0^{-2}	3.28×10.0^{-2}	8.19×10.0^{-4}
		GMM	β_1	0.500	0.238	0.232	5.64×10.0^{-2}
			β_2	0.508	0.198	0.190	3.93×10.0^{-2}
			γ	0.791	4.53×10.0^{-2}	4.36×10.0^{-2}	2.13×10.0^{-3}

Table 4.4: The simulated GQL and GMM estimates for β and γ under the dynamic model (1.4) with true regression parameter values as $\beta_1 = 0.1$, $\beta_2 = 0.5$ and selected true values for γ; when σ_η^2 and σ_ϵ^2 ($\sigma_\epsilon^2 = 1.0$) are known.

				Quantities			
σ_η^2	γ	Method		SM	SSE	ESE	$SMSE$
0.5	-0.8	GQL	β_1	9.97×10.0^{-2}	8.28×10.0^{-2}	8.29×10.0^{-2}	6.86×10.0^{-3}
			β_2	0.501	7.20×10.0^{-2}	7.18×10.0^{-2}	5.18×10.0^{-3}
			γ	-0.797	3.96×10.0^{-2}	2.72×10.0^{-2}	1.57×10.0^{-3}
		GMM	β_1	9.35×10.0^{-2}	0.134	0.129	1.80×10.0^{-2}
			β_2	0.505	0.114	0.118	1.30×10.0^{-2}
			γ	-0.793	4.01×10.0^{-2}	3.91×10.0^{-2}	1.66×10.0^{-3}
	-0.3	GQL	β_1	9.99×10.0^{-2}	8.30×10.0^{-2}	8.29×10.0^{-2}	6.88×10.0^{-3}
			β_2	0.501	7.32×10.0^{-2}	7.18×10.0^{-2}	5.36×10.0^{-3}
			γ	-0.299	4.91×10.0^{-2}	5.29×10.0^{-2}	2.41×10.0^{-3}
		GMM	β_1	9.73×10.0^{-2}	9.28×10.0^{-2}	9.05×10.0^{-2}	8.61×10.0^{-3}
			β_2	0.500	8.49×10.0^{-2}	8.44×10.0^{-2}	7.21×10.0^{-3}
			γ	-0.300	4.84×10.0^{-2}	4.72×10.0^{-2}	2.35×10.0^{-3}
	0.01	GQL	β_1	9.99×10.0^{-2}	8.30×10.0^{-2}	8.29×10.0^{-2}	6.89×10.0^{-3}
			β_2	0.502	7.48×10.0^{-2}	7.18×10.0^{-2}	5.60×10.0^{-3}
			γ	7.00×10.0^{-3}	5.59×10.0^{-2}	6.85×10.0^{-2}	3.14×10.0^{-3}
		GMM	β_1	9.83×10.0^{-2}	0.102	9.91×10.0^{-2}	1.03×10.0^{-2}
			β_2	0.502	8.86×10.0^{-2}	8.49×10.0^{-2}	7.86×10.0^{-3}
			γ	4.88×10.0^{-3}	5.55×10.0^{-2}	5.41×10.0^{-2}	3.11×10.0^{-3}
	0.3	GQL	β_1	9.99×10.0^{-2}	8.31×10.0^{-2}	8.29×10.0^{-2}	6.90×10.0^{-3}
			β_2	0.504	7.57×10.0^{-2}	7.18×10.0^{-2}	5.75×10.0^{-3}
			γ	0.294	5.45×10.0^{-2}	7.05×10.0^{-2}	3.00×10.0^{-3}
		GMM	β_1	9.84×10.0^{-2}	0.124	0.121	1.53×10.0^{-2}
			β_2	0.504	9.74×10.0^{-2}	9.16×10.0^{-2}	9.51×10.0^{-3}
			γ	0.292	5.60×10.0^{-2}	5.43×10.0^{-2}	3.20×10.0^{-3}
	0.8	GQL	β_1	9.99×10.0^{-2}	8.30×10.0^{-2}	8.29×10.0^{-2}	6.88×10.0^{-3}
			β_2	0.503	7.39×10.0^{-2}	7.18×10.0^{-2}	5.47×10.0^{-3}
			γ	0.796	3.34×10.0^{-2}	3.69×10.0^{-2}	1.14×10.0^{-3}
		GMM	β_1	9.63×10.0^{-2}	0.183	0.178	3.33×10.0^{-2}
			β_2	0.506	0.153	0.146	2.34×10.0^{-2}
			γ	0.791	4.47×10.0^{-2}	4.31×10.0^{-2}	2.07×10.0^{-3}
1.0	-0.8	GQL	β_1	0.100	8.56×10.0^{-2}	8.62×10.0^{-2}	7.33×10.0^{-3}
			β_2	0.502	7.47×10.0^{-2}	7.43×10.0^{-2}	5.59×10.0^{-3}
			γ	-0.799	4.44×10.0^{-2}	2.04×10.0^{-2}	1.97×10.0^{-3}
		GMM	β_1	9.34×10.0^{-2}	0.137	0.132	1.88×10.0^{-2}
			β_2	0.505	0.121	0.126	1.47×10.0^{-2}
			γ	-0.794	3.86×10.0^{-2}	3.76×10.0^{-2}	1.53×10.0^{-3}

(Table 4.4 contd....)

σ_η^2	γ	Method		Quantities			
				SM	SSE	ESE	$SMSE$
	-0.3	GQL	β_1	0.100	8.57×10.0^{-2}	8.62×10.0^{-2}	7.35×10.0^{-3}
			β_2	0.502	7.64×10.0^{-2}	7.43×10.0^{-2}	5.84×10.0^{-3}
			γ	-0.301	5.89×10.0^{-2}	6.23×10.0^{-2}	3.47×10.0^{-3}
		GMM	β_1	9.68×10.0^{-2}	0.105	0.103	1.11×10.0^{-2}
			β_2	0.501	0.101	0.101	1.02×10.0^{-2}
			γ	-0.302	5.46×10.0^{-2}	5.29×10.0^{-2}	2.99×10.0^{-3}
	0.01	GQL	β_1	0.100	8.58×10.0^{-2}	8.62×10.0^{-2}	7.35×10.0^{-3}
			β_2	0.503	7.79×10.0^{-2}	7.43×10.0^{-2}	6.08×10.0^{-3}
			γ	4.75×10.0^{-3}	6.36×10.0^{-2}	8.46×10.0^{-2}	4.07×10.0^{-3}
		GMM	β_1	9.78×10.0^{-2}	0.124	0.121	1.54×10.0^{-2}
			β_2	0.502	0.107	0.103	1.14×10.0^{-2}
			γ	2.46×10.0^{-3}	6.27×10.0^{-2}	6.08×10.0^{-2}	3.99×10.0^{-3}
	0.3	GQL	β_1	0.100	8.58×10.0^{-2}	8.62×10.0^{-2}	7.35×10.0^{-3}
			β_2	0.504	7.82×10.0^{-2}	7.43×10.0^{-2}	6.13×10.0^{-3}
			γ	0.294	5.49×10.0^{-2}	7.84×10.0^{-2}	3.06×10.0^{-3}
		GMM	β_1	9.78×10.0^{-2}	0.157	0.154	2.45×10.0^{-2}
			β_2	0.504	0.119	0.113	1.41×10.0^{-2}
			γ	0.290	5.96×10.0^{-2}	5.76×10.0^{-2}	3.65×10.0^{-3}
	0.8	GQL	β_1	0.100	8.56×10.0^{-2}	8.62×10.0^{-2}	7.33×10.0^{-3}
			β_2	0.503	7.56×10.0^{-2}	7.43×10.0^{-2}	5.73×10.0^{-3}
			γ	0.797	2.85×10.0^{-2}	3.28×10.0^{-2}	8.19×10.0^{-4}
		GMM	β_1	9.58×10.0^{-2}	0.237	0.232	5.63×10.0^{-2}
			β_2	0.505	0.196	0.190	3.85×10.0^{-2}
			γ	0.791	4.53×10.0^{-2}	4.36×10.0^{-2}	2.13×10.0^{-3}

produces smaller biases than those under the GMM approach. For example, for the estimation of $\beta_1 = \beta_2 = 0.5$, $\gamma = 0.8$ when $\sigma_\eta^2 = 1.0$ and $\sigma_\epsilon^2 = 1.0$, GQL produces a smaller bias 0.003 for γ as compared with the bias 0.009 by GMM. However, it is clear that the GMM and GQL appear to be very competitive when SSE and SMSE are concerned. For example, for the above selected parameter values, the SSE under GMM is $\frac{4.53\times10.0^{-2}}{2.85\times10.0^{-2}} = 1.590$ times of the SSE under the GQL approach, and the SMSE under GMM is $\frac{2.13\times10.0^{-3}}{8.19\times10.0^{-4}} = 2.601$ times of the SMSE under GQL. For another parameter set where $\beta_1 = \beta_2 = 0.5$, $\gamma = -0.8$, $\sigma_\eta^2 = 1.0$, and $\sigma_\epsilon^2 = 1.0$, the SSE under GQL is $\frac{4.45\times10.0^{-2}}{3.86\times10.0^{-2}} = 1.153$ times of the SSE under the GMM approach, and the SMSE under GQL is $\frac{1.98\times10.0^{-3}}{1.53\times10.0^{-3}} = 1.294$ times of the SMSE under GMM.

As far as the performance of the ESE is concerned, ESEs appear to be close to the corresponding SSEs under both GMM and GQL approaches. For example, for the above selected parameter values where $\gamma = 0.8$, the SSE($\hat{\gamma}_{GQL}$) is 2.85×10.0^{-2}, and the ESE($\hat{\gamma}_{GQL}$) is 3.28×10.0^{-2}, which are very close to each other. Similar results hold for $\hat{\gamma}_{GMM}$.

4.3 Estimation of All Parameters

As all parameters are supposed to be unknown in practice, we now consider this important situation. We estimate β and γ using the equations mentioned in section 4.2, but for the estimation of $\alpha = (\sigma_\eta^2, \sigma_\epsilon^2)'$, we use the iterative equation given by

$$\hat{\alpha}_{GMM,(r+1)} = \hat{\alpha}_{GMM,(r)} + \left[\frac{\partial \psi_3^{*\prime}}{\partial \alpha} C_{3N} \frac{\partial \psi_3^*}{\partial \alpha'}\right]^{-1}_{(r)} \left[\frac{\partial \psi_3^{*\prime}}{\partial \alpha} C_{3N} \psi_3^*\right]_{(r)} \tag{4.5}$$

under the GMM approach; and

$$\hat{\alpha}_{GQL,(r+1)} = \hat{\alpha}_{GQL,(r)} + \left[\sum_{i=1}^{K} \frac{\partial \sigma_i'}{\partial \alpha} \Omega_{iN}^{-1} \frac{\partial \sigma_i}{\partial \alpha'}\right]^{-1}_{(r)} \left[\sum_{i=1}^{K} \frac{\partial \sigma_i'}{\partial \alpha} \Omega_{iN}^{-1} (s_i - \sigma_i)\right]_{(r)} \tag{4.6}$$

under the GQL approach.

The GMM and GQL estimates for all parameters along with their SSE, ESE and SMSE based on 500 simulations are given in Tables 4.5-4.8. The estimates of β and γ

appear to be similar to those given in Tables 4.3-4.4. We now interpret the estimates of α. From these tables, it is clear that GMM and GQL are very competitive in estimating α. For example, for estimation of σ_η^2, for the selected parameter values $\beta_1 = 0.1$, $\beta_2 = 0.5$, $\gamma = -0.8$ with $\sigma_\eta^2 = 0.5$ and $\sigma_\epsilon^2 = 1.0$, GQL produced a bias 0.004, which is smaller than the bias 0.029 by GMM, the SSE($\hat{\sigma}^2_{\eta,GMM}$) is $\frac{0.114}{0.105} = 1.086$ times of the SSE($\hat{\sigma}^2_{\eta,GQL}$), and the corresponding SMSE under GMM is $\frac{1.39\times 10.0^{-2}}{1.09\times 10.0^{-2}} = 1.275$ times of the SMSE under GQL. Another example to show that GQL appears to be better than GMM is for estimation of σ_ϵ^2. For the selected parameter values $\beta_1 = 0.1$, $\beta_2 = 0.5$, $\gamma = 0.8$ with $\sigma_\eta^2 = 1.0$ and $\sigma_\epsilon^2 = 1.0$, GQL produces a smaller bias 0.009 than the bias 0.104 by GMM, the SSE under GMM is $\frac{0.297}{7.64\times 10.0^{-2}} = 3.887$ times of the SSE under GQL, and the SMSE under GMM is $\frac{9.89\times 10.0^{-2}}{5.93\times 10.0^{-3}} = 16.678$ times of the SMSE under GQL. On the other hand, in some cases, GMM appears to be better than GQL. For example, for the selected parameter values $\beta_1 = \beta_2 = 0.5$, $\gamma = 0.3$ with $\sigma_\eta^2 = 1.0$ and $\sigma_\epsilon^2 = 1.0$, for σ_η^2, GMM produced a smaller bias 0.131 compared with the bias 0.299 by GQL, the SSE($\hat{\sigma}^2_{\eta,GQL}$) is $\frac{0.285}{0.106} = 2.689$ times of the SSE($\hat{\sigma}^2_{\eta,GMM}$), and the SMSE under GQL is $\frac{0.171}{2.83\times 10.0^{-2}} = 6.042$ times of the SMSE under GMM. For σ_ϵ^2, GMM produced a bias 0.005 which is smaller than the bias 0.081 by GQL, the SSE under GQL is $\frac{0.160}{6.74\times 10.0^{-2}} = 2.374$ times of the SSE under the GMM approach, and the SMSE under GQL is $\frac{3.21\times 10.0^{-2}}{4.57\times 10.0^{-3}} = 7.024$ times of the SMSE under GMM.

In summary, GQL was found to be better than GMM in estimating the regression effects β_1 and β_2. The approaches were found to be competitive for the estimation of γ, σ_η^2 and σ_ϵ^2. Note however that all these GMM results reported in Tables 4.5 to 4.8 were obtained by using initial values close to the true parameter values, whereas the GQL approach was not sensitive to the selection of the initial values. This is a serious technical limitation for the GMM approach, as in practice, one does not have any idea about the true values. This observation along with the fact that GQL has performed very well for β estimation and it was extremely competitive to GMM for γ and α estimation, suggest that one prefer GQL over the GMM approach.

Table 4.5: The simulated GQL and GMM estimates for all parameters under the dynamic model (1.4), with true $\beta_1 = \beta_2 = 0.5$ and selected true values for γ, $\sigma_\eta^2 = 0.5$ and $\sigma_\epsilon^2 = 1.0$.

γ	Method		Quantities			
			SM	*SSE*	*ESE*	*SMSE*
-0.8	GQL	β_1	0.495	8.48×10.0^{-2}	8.24×10.0^{-2}	7.22×10.0^{-3}
		β_2	0.505	7.15×10.0^{-2}	7.14×10.0^{-2}	5.14×10.0^{-3}
		γ	-0.800	3.82×10.0^{-2}	2.71×10.0^{-2}	1.46×10.0^{-3}
		σ_η^2	0.504	0.105	9.63×10.0^{-2}	1.10×10.0^{-2}
		σ_ϵ^2	0.990	6.30×10.0^{-2}	6.28×10.0^{-2}	4.07×10.0^{-3}
	GMM	β_1	0.480	0.138	0.128	1.94×10.0^{-2}
		β_2	0.509	0.114	0.119	1.30×10.0^{-2}
		γ	-0.817	4.59×10.0^{-2}	3.84×10.0^{-2}	2.39×10.0^{-3}
		σ_η^2	0.529	0.114	0.107	1.39×10.0^{-2}
		σ_ϵ^2	0.938	5.69×10.0^{-2}	9.81×10.0^{-2}	7.02×10.0^{-3}
-0.3	GQL	β_1	0.495	8.50×10.0^{-2}	8.24×10.0^{-2}	7.26×10.0^{-3}
		β_2	0.506	7.32×10.0^{-2}	7.14×10.0^{-2}	5.38×10.0^{-3}
		γ	-0.302	4.95×10.0^{-2}	5.28×10.0^{-2}	2.45×10.0^{-3}
		σ_η^2	0.506	0.111	9.67×10.0^{-2}	1.23×10.0^{-2}
		σ_ϵ^2	0.989	6.34×10.0^{-2}	6.27×10.0^{-2}	4.14×10.0^{-3}
	GMM	β_1	0.492	9.40×10.0^{-2}	9.00×10.0^{-2}	8.91×10.0^{-3}
		β_2	0.502	8.39×10.0^{-2}	8.40×10.0^{-2}	7.04×10.0^{-3}
		γ	-0.298	4.86×10.0^{-2}	4.73×10.0^{-2}	2.37×10.0^{-3}
		σ_η^2	0.497	0.106	9.59×10.0^{-2}	1.12×10.0^{-2}
		σ_ϵ^2	0.992	6.24×10.0^{-2}	6.73×10.0^{-2}	3.95×10.0^{-3}
0.01	GQL	β_1	0.495	8.52×10.0^{-2}	8.24×10.0^{-2}	7.28×10.0^{-3}
		β_2	0.500	7.48×10.0^{-2}	7.14×10.0^{-2}	5.59×10.0^{-3}
		γ	2.24×10.0^{-2}	5.40×10.0^{-2}	6.82×10.0^{-2}	3.07×10.0^{-3}
		σ_η^2	0.487	0.103	9.41×10.0^{-2}	1.07×10.0^{-2}
		σ_ϵ^2	0.995	6.40×10.0^{-2}	6.30×10.0^{-2}	4.12×10.0^{-3}
	GMM	β_1	0.492	0.102	9.85×10.0^{-2}	1.04×10.0^{-2}
		β_2	0.498	8.85×10.0^{-2}	8.40×10.0^{-2}	7.84×10.0^{-3}
		γ	2.15×10.0^{-2}	5.26×10.0^{-2}	5.39×10.0^{-2}	2.90×10.0^{-3}
		σ_η^2	0.483	0.101	9.35×10.0^{-2}	1.06×10.0^{-2}
		σ_ϵ^2	0.997	6.37×10.0^{-2}	6.33×10.0^{-2}	4.07×10.0^{-3}
0.3	GQL	β_1	0.496	8.51×10.0^{-2}	8.24×10.0^{-2}	7.26×10.0^{-3}
		β_2	0.486	7.65×10.0^{-2}	7.15×10.0^{-2}	6.05×10.0^{-3}
		γ	0.335	5.61×10.0^{-2}	6.74×10.0^{-2}	4.37×10.0^{-3}
		σ_η^2	0.448	9.88×10.0^{-2}	8.88×10.0^{-2}	1.25×10.0^{-2}
		σ_ϵ^2	1.008	6.82×10.0^{-2}	6.39×10.0^{-2}	4.71×10.0^{-3}
	GMM	β_1	0.486	0.123	0.119	1.528×10.0^{-2}
		β_2	0.484	9.77×10.0^{-2}	8.98×10.0^{-2}	9.81×10.0^{-3}
		γ	0.333	5.22×10.0^{-2}	5.32×10.0^{-2}	3.81×10.0^{-3}
		σ_η^2	0.446	8.19×10.0^{-2}	8.85×10.0^{-2}	9.65×10.0^{-3}
		σ_ϵ^2	0.997	6.66×10.0^{-2}	6.90×10.0^{-2}	4.45×10.0^{-3}

(Table 4.5 contd....)

γ	Method		Quantities			
			SM	SSE	ESE	$SMSE$
0.8	GQL	β_1	0.495	8.47×10.0^{-2}	8.24×10.0^{-2}	7.20×10.0^{-3}
		β_2	0.494	8.17×10.0^{-2}	7.14×10.0^{-2}	6.71×10.0^{-3}
		γ	0.812	4.94×10.0^{-2}	3.61×10.0^{-2}	2.58×10.0^{-3}
		σ_η^2	0.477	0.149	9.40×10.0^{-2}	2.26×10.0^{-2}
		σ_ϵ^2	1.001	7.31×10.0^{-2}	6.35×10.0^{-2}	5.34×10.0^{-3}
	GMM	β_1	0.469	0.172	0.167	3.05×10.0^{-2}
		β_2	0.456	0.153	0.144	2.52×10.0^{-2}
		γ	0.853	4.49×10.0^{-2}	4.17×10.0^{-2}	4.84×10.0^{-3}
		σ_η^2	0.396	6.17×10.0^{-2}	9.32×10.0^{-2}	1.45×10.0^{-2}
		σ_ϵ^2	0.901	0.207	0.172	5.25×10.0^{-2}

Table 4.6: The simulated GQL and GMM estimates for all parameters under the dynamic model (1.4), with true $\beta_1 = \beta_2 = 0.5$ and selected true values for γ, $\sigma_\eta^2 = 1.0$ and $\sigma_\epsilon^2 = 1.0$.

γ	Method		Quantities			
			SM	*SSE*	*ESE*	*SMSE*
-0.8	GQL	β_1	0.496	8.82×10.0^{-2}	8.58×10.0^{-2}	7.79×10.0^{-3}
		β_2	0.506	7.42×10.0^{-2}	7.39×10.0^{-2}	5.54×10.0^{-3}
		γ	-0.801	4.44×10.0^{-2}	2.05×10.0^{-2}	1.97×10.0^{-3}
		σ_η^2	1.005	0.184	0.168	3.38×10.0^{-2}
		σ_ϵ^2	0.991	6.33×10.0^{-2}	6.28×10.0^{-2}	4.09×10.0^{-3}
	GMM	β_1	0.485	0.141	0.131	2.02×10.0^{-2}
		β_2	0.507	0.120	0.126	1.45×10.0^{-2}
		γ	-0.807	4.58×10.0^{-2}	3.72×10.0^{-2}	2.16×10.0^{-3}
		σ_η^2	1.021	0.188	0.177	3.56×10.0^{-2}
		σ_ϵ^2	0.960	5.94×10.0^{-2}	0.102	5.15×10.0^{-3}
-0.3	GQL	β_1	0.496	8.86×10.0^{-2}	8.60×10.0^{-2}	7.86×10.0^{-3}
		β_2	0.499	7.53×10.0^{-2}	7.41×10.0^{-2}	5.67×10.0^{-3}
		γ	−0.282	5.69×10.0^{-2}	6.27×10.0^{-2}	3.58×10.0^{-3}
		σ_η^2	0.972	0.170	0.163	2.99×10.0^{-2}
		σ_ϵ^2	0.996	6.30×10.0^{-2}	6.31×10.0^{-2}	3.99×10.0^{-3}
	GMM	β_1	0.493	0.107	0.102	1.15×10.0^{-2}
		β_2	0.493	9.86×10.0^{-2}	9.85×10.0^{-2}	9.78×10.0^{-3}
		γ	-0.272	5.02×10.0^{-2}	5.28×10.0^{-2}	3.32×10.0^{-3}
		σ_η^2	0.945	0.162	0.160	2.92×10.0^{-2}
		σ_ϵ^2	1.014	6.26×10.0^{-2}	6.82×10.0^{-2}	4.12×10.0^{-3}
0.01	GQL	β_1	0.499	8.94×10.0^{-2}	8.68×10.0^{-2}	8.00×10.0^{-3}
		β_2	0.466	7.41×10.0^{-2}	7.49×10.0^{-2}	6.62×10.0^{-3}
		γ	0.105	5.69×10.0^{-2}	7.87×10.0^{-2}	1.23×10.0^{-2}
		σ_η^2	0.811	0.130	0.141	5.25×10.0^{-2}
		σ_ϵ^2	1.032	6.71×10.0^{-2}	6.54×10.0^{-2}	5.55×10.0^{-3}
	GMM	β_1	0.490	0.125	0.121	1.56×10.0^{-2}
		β_2	0.492	0.105	0.101	1.11×10.0^{-2}
		γ	3.44×10.0^{-2}	5.08×10.0^{-2}	6.01×10.0^{-2}	3.18×10.0^{-3}
		σ_η^2	0.938	0.147	0.158	2.55×10.0^{-2}
		σ_ϵ^2	1.003	6.38×10.0^{-2}	6.38×10.0^{-2}	4.08×10.0^{-3}
0.3	GQL	β_1	0.498	9.03×10.0^{-2}	8.75×10.0^{-2}	8.16×10.0^{-3}
		β_2	0.438	0.102	7.58×10.0^{-2}	1.43×10.0^{-2}
		γ	0.425	0.122	6.65×10.0^{-2}	3.05×10.0^{-2}
		σ_η^2	0.701	0.285	0.131	0.171
		σ_ϵ^2	1.081	0.160	6.91×10.0^{-2}	3.21×10.0^{-2}
	GMM	β_1	0.483	0.156	0.152	2.46×10.0^{-2}
		β_2	0.476	0.117	0.110	1.42×10.0^{-2}
		γ	0.346	5.04×10.0^{-2}	5.56×10.0^{-2}	4.63×10.0^{-3}
		σ_η^2	0.869	0.106	0.148	2.83×10.0^{-2}
		σ_ϵ^2	0.995	6.74×10.0^{-2}	7.09×10.0^{-2}	4.57×10.0^{-3}

(Table 4.6 contd....)

γ	Method		Quantities			
			SM	SSE	ESE	$SMSE$
0.8	GQL	β_1	0.498	8.85×10.0^{-2}	8.61×10.0^{-2}	7.84×10.0^{-3}
		β_2	0.486	8.42×10.0^{-2}	7.43×10.0^{-2}	7.29×10.0^{-3}
		γ	0.823	4.79×10.0^{-2}	3.22×10.0^{-2}	2.85×10.0^{-3}
		σ_η^2	0.906	0.245	0.156	6.86×10.0^{-2}
		σ_ϵ^2	1.013	9.12×10.0^{-2}	6.43×10.0^{-2}	8.49×10.0^{-3}
	GMM	β_1	0.482	0.231	0.225	5.37×10.0^{-2}
		β_2	0.480	0.196	0.189	3.89×10.0^{-2}
		γ	0.825	4.42×10.0^{-2}	4.29×10.0^{-2}	2.56×10.0^{-3}
		σ_η^2	0.902	8.78×10.0^{-2}	0.161	1.73×10.0^{-2}
		σ_ϵ^2	0.897	0.297	0.195	9.86×10.0^{-2}

Table 4.7: The simulated GQL and GMM estimates for all parameters under the dynamic model (1.4), with true $\beta_1 = 0.1$, $\beta_2 = 0.5$ and selected true values for γ, $\sigma_\eta^2 = 0.5$ and $\sigma_\epsilon^2 = 1.0$.

			Quantities			
γ	Method		SM	SSE	ESE	$SMSE$
-0.8	GQL	β_1	9.52×10.0^{-2}	8.48×10.0^{-2}	8.24×10.0^{-2}	7.22×10.0^{-3}
		β_2	0.505	7.14×10.0^{-2}	7.14×10.0^{-2}	5.12×10.0^{-3}
		γ	-0.800	3.82×10.0^{-2}	2.71×10.0^{-2}	1.46×10.0^{-3}
		σ_η^2	0.504	0.105	9.63×10.0^{-2}	1.09×10.0^{-2}
		σ_ϵ^2	0.990	6.30×10.0^{-2}	6.28×10.0^{-2}	4.07×10.0^{-3}
	GMM	β_1	8.88×10.0^{-2}	0.135	0.128	1.83×10.0^{-2}
		β_2	0.507	0.113	0.119	1.29×10.0^{-2}
		γ	-0.817	4.60×10.0^{-2}	3.84×10.0^{-2}	2.34×10.0^{-3}
		σ_η^2	0.529	0.114	0.107	1.39×10.0^{-2}
		σ_ϵ^2	0.938	5.71×10.0^{-2}	9.80×10.0^{-2}	7.11×10.0^{-3}
-0.3	GQL	β_1	9.53×10.0^{-2}	8.51×10.0^{-2}	8.24×10.0^{-2}	7.26×10.0^{-3}
		β_2	0.506	7.26×10.0^{-2}	7.13×10.0^{-2}	5.31×10.0^{-3}
		γ	-0.302	4.95×10.0^{-2}	5.28×10.0^{-2}	2.46×10.0^{-3}
		σ_η^2	0.506	0.111	9.67×10.0^{-2}	1.23×10.0^{-2}
		σ_ϵ^2	0.989	6.34×10.0^{-2}	6.27×10.0^{-2}	4.14×10.0^{-3}
	GMM	β_1	9.29×10.0^{-2}	9.39×10.0^{-2}	9.00×10.0^{-2}	8.88×10.0^{-3}
		β_2	0.502	8.30×10.0^{-2}	8.40×10.0^{-2}	6.89×10.0^{-3}
		γ	-0.298	4.87×10.0^{-2}	4.73×10.0^{-2}	2.38×10.0^{-3}
		σ_η^2	0.497	0.106	9.59×10.0^{-2}	1.12×10.0^{-2}
		σ_ϵ^2	0.992	6.24×10.0^{-2}	6.73×10.0^{-2}	3.96×10.0^{-3}
0.01	GQL	β_1	9.56×10.0^{-2}	8.52×10.0^{-2}	8.24×10.0^{-2}	7.28×10.0^{-3}
		β_2	0.501	7.34×10.0^{-2}	7.14×10.0^{-2}	5.38×10.0^{-3}
		γ	2.25×10.0^{-2}	5.41×10.0^{-2}	6.82×10.0^{-2}	3.08×10.0^{-3}
		σ_η^2	0.487	0.103	9.41×10.0^{-2}	1.07×10.0^{-2}
		σ_ϵ^2	0.994	6.40×10.0^{-2}	6.30×10.0^{-2}	4.12×10.0^{-3}
	GMM	β_1	9.37×10.0^{-2}	0.102	9.85×10.0^{-2}	1.04×10.0^{-2}
		β_2	0.499	8.68×10.0^{-2}	8.40×10.0^{-2}	7.54×10.0^{-3}
		γ	2.17×10.0^{-2}	5.26×10.0^{-2}	5.39×10.0^{-2}	2.91×10.0^{-3}
		σ_η^2	0.483	0.101	9.35×10.0^{-2}	1.06×10.0^{-2}
		σ_ϵ^2	0.997	6.37×10.0^{-2}	6.33×10.0^{-2}	4.07×10.0^{-3}
0.3	GQL	β_1	9.62×10.0^{-2}	8.51×10.0^{-2}	8.24×10.0^{-2}	7.26×10.0^{-3}
		β_2	0.491	7.35×10.0^{-2}	7.15×10.0^{-2}	5.49×10.0^{-3}
		γ	0.335	5.59×10.0^{-2}	6.74×10.0^{-2}	4.37×10.0^{-3}
		σ_η^2	0.448	9.80×10.0^{-2}	8.87×10.0^{-2}	1.23×10.0^{-2}
		σ_ϵ^2	1.008	6.80×10.0^{-2}	6.39×10.0^{-2}	4.69×10.0^{-3}
	GMM	β_1	9.40×10.0^{-2}	0.123	0.119	1.51×10.0^{-2}
		β_2	0.488	9.52×10.0^{-2}	8.98×10.0^{-2}	9.19×10.0^{-3}
		γ	0.333	5.22×10.0^{-2}	5.32×10.0^{-2}	3.82×10.0^{-3}
		σ_η^2	0.446	8.18×10.0^{-2}	8.85×10.0^{-2}	9.65×10.0^{-3}
		σ_ϵ^2	0.996	6.66×10.0^{-2}	6.90×10.0^{-2}	4.45×10.0^{-3}

(Table 4.7 contd....)

γ	Method		Quantities			
			SM	SSE	ESE	$SMSE$
0.8	GQL	β_1	9.54×10.0^{-2}	8.46×10.0^{-2}	8.24×10.0^{-2}	7.18×10.0^{-3}
		β_2	0.499	7.45×10.0^{-2}	7.14×10.0^{-2}	5.55×10.0^{-3}
		γ	0.812	4.92×10.0^{-2}	3.62×10.0^{-2}	2.55×10.0^{-3}
		σ_η^2	0.477	0.147	9.39×10.0^{-2}	2.22×10.0^{-2}
		σ_ϵ^2	1.001	7.29×10.0^{-2}	6.35×10.0^{-2}	5.32×10.0^{-3}
	GMM	β_1	9.80×10.0^{-2}	0.172	0.167	2.97×10.0^{-2}
		β_2	0.465	0.151	0.144	2.39×10.0^{-2}
		γ	0.853	4.49×10.0^{-2}	4.17×10.0^{-2}	4.84×10.0^{-3}
		σ_η^2	0.397	6.18×10.0^{-2}	9.32×10.0^{-2}	1.45×10.0^{-2}
		σ_ϵ^2	0.900	0.207	0.172	5.28×10.0^{-2}

Table 4.8: The simulated GQL and GMM estimates for all parameters under the dynamic model (1.4), with true $\beta_1 = 0.1$, $\beta_2 = 0.5$ and selected true values for γ, $\sigma^2_\eta = 1.0$ and $\sigma^2_\epsilon = 1.0$.

			Quantities			
γ	Method		*SM*	*SSE*	*ESE*	*SMSE*
-0.8	GQL	β_1	9.59×10.0^{-2}	8.82×10.0^{-2}	8.58×10.0^{-2}	7.79×10.0^{-3}
		β_2	0.506	7.41×10.0^{-2}	7.39×10.0^{-2}	5.52×10.0^{-3}
		γ	-0.801	4.44×10.0^{-2}	2.05×10.0^{-2}	1.97×10.0^{-3}
		σ^2_η	1.005	0.184	0.168	3.38×10.0^{-2}
		σ^2_ϵ	0.991	6.33×10.0^{-2}	6.28×10.0^{-2}	4.08×10.0^{-3}
	GMM	β_1	8.93×10.0^{-2}	0.138	0.131	1.92×10.0^{-2}
		β_2	0.506	0.120	0.126	1.44×10.0^{-2}
		γ	-0.807	4.59×10.0^{-2}	3.72×10.0^{-2}	2.16×10.0^{-3}
		σ^2_η	1.021	0.188	0.177	3.57×10.0^{-2}
		σ^2_ϵ	0.959	5.96×10.0^{-2}	0.102	5.20×10.0^{-3}
-0.3	GQL	β_1	9.61×10.0^{-2}	8.85×10.0^{-2}	8.59×10.0^{-2}	7.85×10.0^{-3}
		β_2	0.500	7.48×10.0^{-2}	7.41×10.0^{-2}	5.59×10.0^{-3}
		γ	-0.282	5.73×10.0^{-2}	6.27×10.0^{-2}	3.62×10.0^{-3}
		σ^2_η	0.971	0.171	0.163	2.99×10.0^{-2}
		σ^2_ϵ	0.995	6.30×10.0^{-2}	6.31×10.0^{-2}	3.99×10.0^{-3}
	GMM	β_1	9.28×10.0^{-2}	0.107	0.102	1.15×10.0^{-2}
		β_2	0.495	9.78×10.0^{-2}	9.85×10.0^{-2}	9.58×10.0^{-3}
		γ	-0.272	5.05×10.0^{-2}	5.28×10.0^{-2}	3.35×10.0^{-3}
		σ^2_η	0.945	0.162	0.160	2.92×10.0^{-2}
		σ^2_ϵ	1.014	6.27×10.0^{-2}	6.82×10.0^{-2}	4.12×10.0^{-3}
0.01	GQL	β_1	9.78×10.0^{-2}	8.94×10.0^{-2}	8.67×10.0^{-2}	8.00×10.0^{-3}
		β_2	0.473	7.26×10.0^{-2}	7.49×10.0^{-2}	5.99×10.0^{-3}
		γ	0.105	5.71×10.0^{-2}	7.88×10.0^{-2}	1.22×10.0^{-2}
		σ^2_η	0.811	0.131	0.141	5.27×10.0^{-2}
		σ^2_ϵ	1.03	6.70×10.0^{-2}	6.54×10.0^{-2}	5.49×10.0^{-3}
	GMM	β_1	9.37×10.0^{-2}	0.125	0.121	1.56×10.0^{-2}
		β_2	0.498	0.104	0.102	1.08×10.0^{-2}
		γ	2.25×10.0^{-2}	4.94×10.0^{-2}	6.05×10.0^{-2}	2.59×10.0^{-3}
		σ^2_η	0.962	0.154	0.161	2.50×10.0^{-2}
		σ^2_ϵ	0.999	6.38×10.0^{-2}	6.35×10.0^{-2}	4.07×10.0^{-3}
0.3	GQL	β_1	9.94×10.0^{-2}	8.98×10.0^{-2}	8.67×10.0^{-2}	8.06×10.0^{-3}
		β_2	0.464	7.75×10.0^{-2}	7.50×10.0^{-2}	7.33×10.0^{-3}
		γ	0.408	7.76×10.0^{-2}	6.78×10.0^{-2}	1.76×10.0^{-2}
		σ^2_η	0.734	0.181	0.131	0.104
		σ^2_ϵ	1.055	0.105	6.71×10.0^{-2}	1.41×10.0^{-2}
	GMM	β_1	9.40×10.0^{-2}	0.157	0.153	2.45×10.0^{-2}
		β_2	0.493	0.116	0.112	1.35×10.0^{-2}
		γ	0.322	4.88×10.0^{-2}	5.65×10.0^{-2}	2.86×10.0^{-3}
		σ^2_η	0.926	0.114	0.156	1.86×10.0^{-2}
		σ^2_ϵ	0.998	6.70×10.0^{-2}	7.00×10.0^{-2}	4.50×10.0^{-3}

(Table 4.8 contd....)

γ	Method		Quantities			
			SM	SSE	ESE	$SMSE$
0.8	GQL	β_1	9.70×10.0^{-2}	8.84×10.0^{-2}	8.61×10.0^{-2}	7.82×10.0^{-3}
		β_2	0.495	7.60×10.0^{-2}	7.42×10.0^{-2}	5.79×10.0^{-3}
		γ	0.822	4.16×10.0^{-2}	3.22×10.0^{-2}	2.19×10.0^{-3}
		σ_η^2	0.913	0.230	0.157	6.05×10.0^{-2}
		σ_ϵ^2	1.009	7.64×10.0^{-2}	6.40×10.0^{-2}	5.93×10.0^{-3}
	GMM	β_1	9.54×10.0^{-2}	0.232	0.225	5.37×10.0^{-2}
		β_2	0.484	0.194	0.189	3.81×10.0^{-2}
		γ	0.825	4.42×10.0^{-2}	4.29×10.0^{-2}	2.56×10.0^{-3}
		σ_η^2	0.902	8.79×10.0^{-2}	0.161	1.74×10.0^{-2}
		σ_ϵ^2	0.896	0.297	0.195	9.89×10.0^{-2}

Chapter 5

Concluding Remarks

In this thesis, we have provided two inference techniques to analyze continuous panel data by fitting a linear dynamic mixed model with dynamic mean structure. Our interest is to estimate the model parameters, namely, the regression parameter (β), the dynamic dependence parameter (γ), the variances of the individual effects (σ_η^2) and the variance of random error terms (σ_ϵ^2). We explained some existing estimation methods briefly in chapter 2, such as the least squares dummy variable (LSDV), bias-corrected LSDV, and instrumental variables based generalized method of moments (IVGMM). The proposed two inference techniques, namely, the GMM as well as a generalized quasi-likelihood estimating approach (GQL), are given in the same chapter. Note that some of these existing estimation methods may involve either first differencing the dynamic panel data model or subtracting the individual-specific means, which eliminates the unobservable individual specific effects during the process. Therefore, it is impossible to estimate the variance of the individual specific effects. Unlike these existing methods, our new GMM and GQL estimation methods, although based on instrumental variables as well, however, performed well to estimate the variance component of the individual effects.

In chapter 3, we compared the asymptotic efficiency of the proposed GMM and GQL estimation methods. It was demonstrated that the GQL outperforms GMM uniformly for the estimation of the parameters β, σ_η^2 and σ_ϵ^2. For the estimation of the dynamic dependence parameter γ, the GMM approach was found to perform better than the GQL approach in a few cases.

In chapter 4, we examined the relative performances of the GQL and GMM approaches through a small sample simulation study. It was found that the GQL outperforms GMM for the estimation of the regression parameter β, but the two approaches appear to be very competitive for the estimation of the other parameters, γ, σ_η^2, and σ_ϵ^2. However, we found that the GMM has a serious technical limitation as its iterative equations required the initial values to be very close to the true parameter values to yield converged estimates. This is however impractical, as in practice, it is difficult to consider such initial values.

In summary, although the GMM estimation approach is widely used in estimating the parameters of the linear panel data models, it is however demonstrated in the thesis that the GQL estimation approach is a more efficient and reliable approach than the GMM approach. Consequently, we recommend the GQL approach in practice for the analysis of the linear dynamic mixed models based panel data.

Appendix A

Formulas for the partial derivatives : The following formulas are used to compute the partial derivatives needed for the GMM, GQL and GQL* estimating equations.

$$\frac{\partial\psi_1}{\partial\beta} = \sum_{i=1}^{K}\sum_{t=1}^{T}(\sum_{j=0}^{t-1}\gamma^j x_{i,t-j})(\sum_{j=0}^{t-1}\gamma^j x'_{i,t-j}), \tag{a.1}$$

$$\frac{\partial\psi_2}{\partial\beta} = 0, \tag{a.2}$$

$$\begin{aligned}\frac{\partial\psi_1}{\partial\gamma} &= \sum_{i=1}^{K}\sum_{t=1}^{T-1}(\sum_{j=1}^{t-1} j\gamma^{j-1}x'_{i,t-j}\beta\sum_{j=0}^{t-1}\gamma^j x_{i,t-j} \\ &+\sum_{j=0}^{t-1}\gamma^j x'_{i,t-j}\beta\sum_{j=1}^{t-1}j\gamma^{j-1}x_{i,t-j}),\end{aligned} \tag{a.3}$$

$$\begin{aligned}\frac{\partial\psi_2}{\partial\gamma} &= \sum_{i=1}^{K}\sum_{t=1}^{T-1}[\sigma_\eta^2(\sum_{j=0}^{t-1}\gamma^j\sum_{k=1}^{t}k\gamma^{k-1}+\sum_{j=1}^{t-1}j\gamma^{j-1}\sum_{k=0}^{t}\gamma^k) \\ &+\sigma_\epsilon^2\sum_{j=0}^{t-1}(1+2j)\gamma^{2j}],\end{aligned} \tag{a.4}$$

$$\frac{\partial\psi_3}{\partial\sigma_\eta^2} = K\sum_{t=1}^{T}(\sum_{j=0}^{t-1}\gamma^j)^2, \tag{a.5}$$

$$\frac{\partial\psi_4}{\partial\sigma_\eta^2} = K\sum_{u=1}^{T-1}\sum_{t=u+1}^{T}(\sum_{k=0}^{u-1}\gamma^k)(\sum_{j=0}^{t-1}\gamma^j), \tag{a.6}$$

$$\frac{\partial\psi_3}{\partial\sigma_\epsilon^2} = K\sum_{t=1}^{T}\sum_{j=0}^{t-1}\gamma^{2j}, \tag{a.7}$$

$$\frac{\partial\psi_4}{\partial\sigma_\epsilon^2} = K\sum_{u=1}^{T-1}\sum_{t=u+1}^{T}\sum_{j=0}^{u-1}\gamma^{t-u+2j}, \tag{a.8}$$

$$\frac{\partial\mu_{it}}{\partial\beta} = \sum_{j=0}^{t-1}\gamma^j x_{i,t-j}, \tag{a.9}$$

$$\begin{aligned}\frac{\partial\sigma_{it,t+1}}{\partial\gamma} &= \sigma_\eta^2(\sum_{j=1}^{t-1}j\gamma^{j-1}\sum_{k=0}^{t}\gamma^k+\sum_{j=0}^{t-1}\gamma^j\sum_{k=1}^{t}k\gamma^{k-1}) \\ &+\sigma_\epsilon^2\sum_{j=0}^{t-1}(1+2j)\gamma^{2j},\end{aligned} \tag{a.10}$$

$$\frac{\partial \sigma_{i12}}{\partial \gamma} = \sigma_\eta^2 + \sigma_\epsilon^2, \tag{a.11}$$

$$\frac{\partial \sigma_{itt}}{\partial \sigma_\eta^2} = (\sum_{j=0}^{t-1} \gamma^j)^2, \ \frac{\partial \sigma_{iut}}{\partial \sigma_\eta^2} = \sum_{j=0}^{t-1} \gamma^j \sum_{k=0}^{u-1} \gamma^k, \tag{a.12}$$

$$\frac{\partial \sigma_{itt}}{\partial \sigma_\epsilon^2} = \sum_{j=0}^{t-1} \gamma^{2j}, \ \frac{\partial \sigma_{iut}}{\partial \sigma_\epsilon^2} = \sum_{j=0}^{u-1} \gamma^{t-u+2j}, \tag{a.13}$$

$$\frac{\partial \lambda_{it,t+1}}{\partial \gamma} = \frac{\partial \sigma_{it,t+1}}{\partial \gamma} + \mu_{it} \sum_{j=1}^{t} j\gamma^{j-1} x'_{i,t+1-j}\beta + \mu_{i,t+1} \sum_{j=1}^{t-1} j\gamma^{j-1} x'_{i,t-j}\beta, \tag{a.14}$$

$$\frac{\partial \lambda_{itt}}{\partial \sigma_\eta^2} = \frac{\partial \sigma_{itt}}{\partial \sigma_\eta^2}, \ \frac{\partial \lambda_{iut}}{\partial \sigma_\eta^2} = \frac{\partial \sigma_{iut}}{\partial \sigma_\eta^2}, \tag{a.15}$$

$$\frac{\partial \lambda_{itt}}{\partial \sigma_\epsilon^2} = \frac{\partial \sigma_{itt}}{\partial \sigma_\epsilon^2}, \ \frac{\partial \lambda_{iut}}{\partial \sigma_\epsilon^2} = \frac{\partial \sigma_{iut}}{\partial \sigma_\epsilon^2}. \tag{a.16}$$

Appendix B

Formulas for Variances and Covariances : The following are used to compute the variances and covariances needed for normality based GMM, GQL and GQL* estimating equations.

Under the normality assumption, the following two formulas

$$E\left[(Y_{iu}-\mu_{iu})(Y_{i\ell}-\mu_{i\ell})(Y_{it}-\mu_{it})\right]=0, \tag{b.1}$$

and

$$E\left[(Y_{iu}-\mu_{iu})(Y_{i\ell}-\mu_{i\ell})(Y_{im}-\mu_{im})(Y_{it}-\mu_{it})\right]=\sigma_{iu\ell}\sigma_{imt}+\sigma_{ium}\sigma_{i\ell t}+\sigma_{iut}\sigma_{i\ell m}, \tag{b.2}$$

are used to give necessary third and fourth order moments.

$$\begin{aligned}
\mathrm{var}(\psi_1) &= \sum_{i=1}^{K}\sum_{u=1}^{T}\sum_{t=1}^{T}(\sum_{j=0}^{u-1}\gamma^j x_{i,u-j})\sigma_{iut}(\sum_{j=0}^{t-1}\gamma^j x'_{i,t-j}) && \text{(b.3)}\\
\mathrm{var}(\psi_2) &= \sum_{i=1}^{K}\sum_{u=1}^{T-1}\sum_{t=1}^{T-1}(\sigma_{iut}\sigma_{iu+1,t+1}+\sigma_{iu,t+1}\sigma_{iu+1,t}) && \text{(b.4)}\\
\mathrm{cov}(\psi_1,\psi_2) &= \sum_{i=1}^{K}\sum_{u=1}^{T-1}\sum_{t=1}^{T-1}\mu_{it}\sigma_{iu,u+1}(\sum_{j=0}^{t-1}\gamma^j x_{i,t-j}) \\
&\quad -E(W_1)E(W_2) && \text{(b.5)}\\
\mathrm{var}(\psi_3) &= \sum_{i=1}^{K}[2\sum_{u=1}^{T-1}\sum_{t=u+1}^{T}(\sigma_{iuu}\sigma_{itt}+2\sigma_{iut}^2)+3\sum_{t=1}^{T}\sigma_{itt}^2 \\
&\quad -(\sum_{t=1}^{T}\sigma_{itt})^2] && \text{(b.6)}\\
\mathrm{var}(\psi_4) &= \sum_{i=1}^{K}[\sum_{u=1}^{T-1}\sum_{t=u+1}^{T}\sum_{l=1}^{T-1}\sum_{m=l+1}^{T}(\sigma_{iut}\sigma_{iml}+\sigma_{iul}\sigma_{itm}+\sigma_{ium}\sigma_{itl}) \\
&\quad -(\sum_{u=1}^{T-1}\sum_{t=u+1}^{T}\sigma_{iut})^2] && \text{(b.7)}\\
\mathrm{cov}(\psi_3,\psi_4) &= \sum_{i=1}^{K}[\sum_{t=1}^{T}\sum_{u=1}^{T-1}\sum_{l=u+1}^{T}(\sigma_{iul}\sigma_{itt}+2\sigma_{iut}\sigma_{itl}) \\
&\quad -(\sum_{t=1}^{T}\sigma_{itt})(\sum_{u=1}^{T-1}\sum_{l=u+1}^{T}\sigma_{iul})] && \text{(b.8)}
\end{aligned}$$

$$\begin{aligned}
\mathrm{var}[(y_{it} &- \mu_{it})(y_{i,t+1} - \mu_{i,t+1})] = \sigma^2_{it,t+1} + \sigma_{itt}\sigma_{it,t+1} && \text{(b.9)}\\
\mathrm{cov}[(y_{iu} &- \mu_{iu})(y_{i,u+1} - \mu_{i,u+1}), (y_{it} - \mu_{it})(y_{i,t+1} - \mu_{i,t+1})] = \sigma_{iut}\sigma_{iu+1,t+1} \\
& +\sigma_{iu,t+1}\sigma_{it,u+1} && \text{(b.10)}\\
\mathrm{var}[(y_{it} &- \mu_{it})^2] = 2\sigma^2_{itt} && \text{(b.11)}\\
\mathrm{var}[(y_{iu} &- \mu_{iu})(y_{it} - \mu_{it})] = \sigma^2_{iut} + \sigma_{iuu}\sigma_{itt} && \text{(b.12)}\\
\mathrm{cov}[(y_{iu} &- \mu_{iu})^2, (y_{it} - \mu_{it})^2] = 2\sigma^2_{iut} && \text{(b.13)}\\
\mathrm{cov}[(y_{iu} &- \mu_{iu})^2, (y_{it} - \mu_{it})(y_{il} - \mu_{il})] = 2\sigma_{iut}\sigma_{iul} && \text{(b.14)}\\
\mathrm{cov}[(y_{iu} &- \mu_{iu})(y_{it} - \mu_{it}), (y_{il} - \mu_{il})(y_{im} - \mu_{im})] = \sigma_{iul}\sigma_{itm} + \sigma_{ium}\sigma_{itl} \\
& && \text{(b.15)}\\
\mathrm{var}(y_{it}y_{i,t+1}) &= \sigma^2_{it,t+1} + \sigma_{itt}\sigma_{it+1,t+1} + \mu^2_{it}\sigma_{it+1,t+1} + \mu^2_{it+1}\sigma_{itt} + 2\mu_{it}\mu_{it+1}\sigma_{it,t+1} \\
& && \text{(b.16)}\\
\mathrm{cov}(y_{iu}y_{i,u+1}, y_{it}y_{i,t+1}) &= \sigma_{iut}\sigma_{iu+1,t+1} + \sigma_{iu,t+1}\sigma_{it,u+1} + \mu_{iu}\mu_{it+1}\sigma_{it,u+1} \\
& +\mu_{iu}\mu_{it}\sigma_{iu+1,t+1} + \mu_{iu+1}\mu_{it}\sigma_{iu,t+1} + \mu_{iu+1}\mu_{it+1}\sigma_{iut} && \text{(b.17)}\\
\mathrm{var}(y^2_{it}) &= 2\sigma^2_{itt} + 4\mu^2_{it}\sigma_{itt} && \text{(b.18)}\\
\mathrm{var}(y_{iu}y_{it}) &= \sigma^2_{iut} + \sigma_{iuu}\sigma_{itt} + \mu^2_{iu}\sigma_{itt} + \mu^2_{it}\sigma_{iuu} + 2\mu_{iu}\mu_{it}\sigma_{iut} && \text{(b.19)}\\
\mathrm{cov}(y^2_{iu}, y^2_{it}) &= 2\sigma^2_{iut} + 4\mu_{iu}\mu_{it}\sigma_{iut} && \text{(b.20)}\\
\mathrm{cov}(y^2_{iu}, y_{it}y_{il}) &= 2\sigma_{iut}\sigma_{iul} + 2\mu_{iu}\mu_{it}\sigma_{iul} + 2\mu_{iu}\mu_{il}\sigma_{iut} && \text{(b.21)}\\
\mathrm{cov}(y_{iu}y_{it}, y_{il}y_{im}) &= \sigma_{iul}\sigma_{itm} + \sigma_{ium}\sigma_{itl} \\
& +\mu_{iu}\mu_{im}\sigma_{itl} + \mu_{iu}\mu_{il}\sigma_{itm} + \mu_{it}\mu_{il}\sigma_{ium} + \mu_{it}\mu_{im}\sigma_{iul} && \text{(b.22)}
\end{aligned}$$

Bibliography

[1] Amemiya, T., and Macurdy, T. E. (1986). Instrumental-Variable Estimation of An Error-Components Model. *Econometrica*, **54**, 869-880.

[2] Anderson, T. M., and Hsiao C. (1981). Estimation of Dynamic Models With Error Components. *Journal of American Statistical Association*, **76**, 598-606.

[3] Arellano, M., and Bover, O. (1995). Another Look at the Instrumental Variable Estimatior of Error-Components Models. *Journal of Econometrics*, **68**, 29-52.

[4] Arellano, M., and Bond, S. (1991). Some Tests of Specification for Panel Data: Monte Carlo Evidence and an Application to Employment Equations. *Review of Economics and Statistics*, **58**, 277-298.

[5] Bun, M. J. G., and Carree M. A. (2005). Bias-Corrected Estimation in Dynamic panel Data Models. *Journal of Business and Economic Statistics*, **23**, 200-210.

[6] Hansen, L.-P. (1982). Large-Sample Properties of Generalized Method of Moment Estimators. *Econometrica*, **50**, 1029-1054.

[7] Hsiao, C. (2003). *Analysis of Panel Data*, 2nd Edition. Cambridge University Press.

[8] McCullagh, P. (1983). Quasilikelihood Functions. *Annals of Statistics*, **11**, 59-67.

[9] Rao, R. P., Sutradhar, B. C., and Pandit, V.N. (2008). GMM Versus GQL Inferences in Linear Dynamic Panel Data Models. *unpublished manuscript.*

[10] Sutradhar, B. C. (2003). An Review on Regression Models for Discrete Longitudinal Responses. *Statistical Science*, **18**, 377-393.

[11] Sutradhar, B. C. (2004). On Exact Quasilikelihood Inference in Generalized Linear Mixed Models. *Sankhya: The Indian Journal of Statistics*, **66**, 261-289.

[12] Wedderburn, R. (1974). Quasilikelihood Functions, Generalized Linear Models, and the Gauss-Newton Method. *Biometrika*, **61**, 439-447.